ADVANCED CONSTRUCTIVE GEOMETRY

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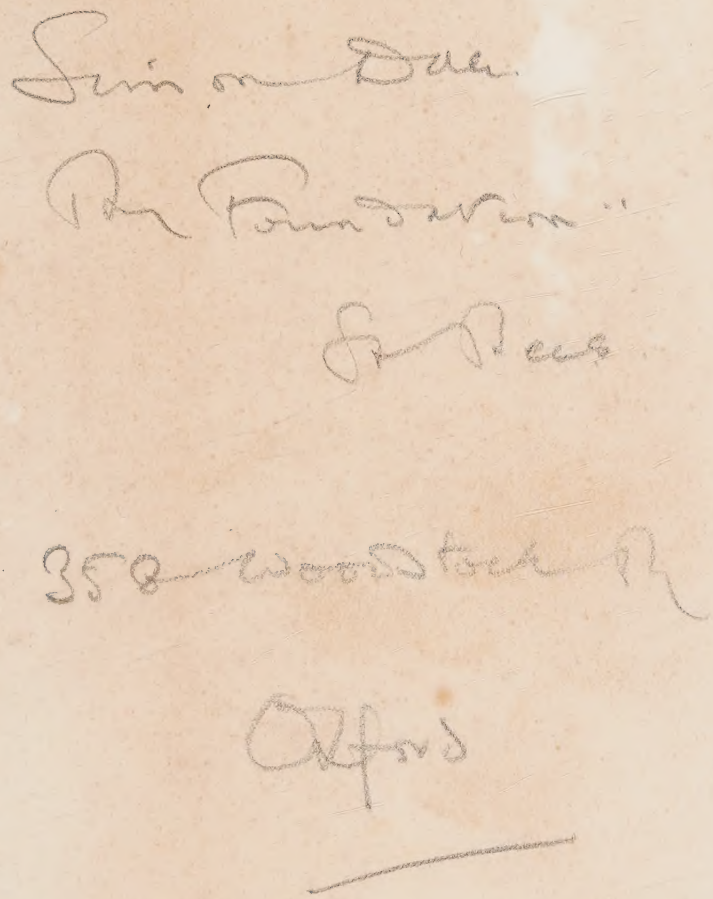

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ADVANCED CONSTRUCTIVE GEOMETRY

ADVANCED CONSTRUCTIVE GEOMETRY

BY

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PREFACE

If any justification is needed for the publication of this work, it is hoped that it will be found in the originality of the matter and the treatment displayed in most of the chapters, particularly in those devoted to practical examples. There has been no intention of following the beaten track, and direct usefulness has been the aim throughout.

The book is intended for advanced students and for teachers of Building Geometry, but fundamental principles are introduced in the demonstrations following the more involved problems.

In keeping the size of the work within reasonable limits, nothing beyond a purely graphical treatment was possible; but since this accords with the requirements of the majority of building students, and a large number of books dealing with the purely quantitative side of the subject already exist, the value is little impaired.

All the geometrical principles involved in the practical examples are dealt with separately, and a caution is given against the very common fault of attempting the application of a geometrical principle before the principle itself has been mastered. The result of thus putting the "cart before the horse" is that the student is led to regard a great many geometrically identical problems as strictly separate constructions, particularly when the materials involved are different, and he acquires little capacity for making the most of his geometrical knowledge.

In the study of geometrical principles time and opportunity only should set the limit when the student's avocation involves the solution of the various geometrical problems arising out of building processes; for no text-book will ever be devised that will contain all the forms in which the problems may be presented, and no principle is too advanced to be outside the range of practical application.

The author hopes to publish an introductory course to this volume,

and also a supplementary volume to amplify the sections that limitations of space have caused to be inadequately dealt with in the present volume.

For convenience in using the book, certain of the illustrations have been duplicated on folders: such figures are marked with an asterisk.

The author is indebted to Mr. A. R. Sage, Principal of the L.C.C. School of Building, for advice and guidance given in connection with the arrangement of the work. Also to Mr. C. Picton, A.R.I.B.A., and to Mr. A. R. H. Jackson, A.R.C.A., for assistance in the preparation of the diagrams.

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SECTION I

PLANE GEOMETRY

CHAPTER I

DEFINITIONS : APPROXIMATE SOLUTIONS

Points which lie in the same straight line are said to be collinear.

Points which lie in the circumference of the same circle are said to be concyclic.

Straight lines which pass through the same point are said to be concurrent.

A line may be considered as the path or locus of a moving point.

A straight line may be considered as the locus of a point which moves without change of direction.

A curve may be considered as the locus of a point moving with constantly changing direction.

The circumference of a circle may be regarded as the locus of a point which moves in a plane with a constant rate of change of direction.

A straight line may be regarded as part of the circumference of a circle of infinitely large radius.

Parallel lines may be considered as lines meeting at infinity.

The measure of an angle is the measure of the difference in the directions of two straight lines.

Two figures are said to be similar when the corresponding angles are equal, and the ratio between any two sides in one of the figures is equal to the ratio between the two corresponding sides in the other figure.

Figures are said to be congruent when they are equal in every respect.

Two lines which continually approach one another but never meet, however far produced, are said to be asymptotic. When one of the lines is straight, the other is a curve, and the straight line is an asymptote of the curve. When both lines are curved, each is an asymptote of the other.

Illustration. *Fig.* 1.—BB^5 is a straight line, P is any point, not on the line. The straight line BAP, on which the distance BA is constant, moving with B always in BB^5 and passing always through P,

will have the curve AA^5 as the locus of A. It will be seen that, as B moves away from P the angle between the two straight lines becomes smaller, and the point A approaches the line in which B is moving; but since the direction of BA is always through P, there will always be an angle between the two straight lines, and the locus of A will never meet the path of B; then BB^5 and AA^5 are asymptotic. The curve $AA^1 \ldots A^5$ is known as the conchoid of Nicomedes. The curve $CC^1 \ldots C^4$ is a second conchoid to which BB^5 is asymptotic; therefore each curve is an asymptote of the other. This curve is sometimes used as the entasis of architectural columns.

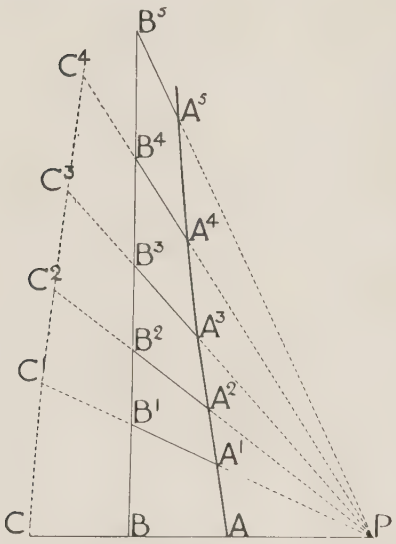


Fig. 1.

Approximate Solutions. — A tangent to a curve at a given point can only be determined accurately by construction, when the exact nature of the curve is known; for undefined curves, such as those drawn freehand, only approximate solutions are available.

Illustration. Fig. 2.—MN is any given curve and P is any given point in the curve; it is required to draw a tangent at P. Then if Pa equals Pb, and Pc equals Pd, the chords ab and cd will be parallel when the curvatures of the arcs Pbd and Pac are equal, and the tangent at P will be parallel to the chords. When the curvatures are unequal, as in the given example, the chords will converge towards the side of lesser curvature, the degree of convergence increasing as the difference in the curvatures increases. If the required tangent is considered as a chord which cuts the arc in two points infinitely near to P, it will pass through the point of intersection of the two chords ab and cd, produced, *if the distances Pa, Pb, Pc, and Pd are made sufficiently small* and if there is no violent change in the nature of the curve between

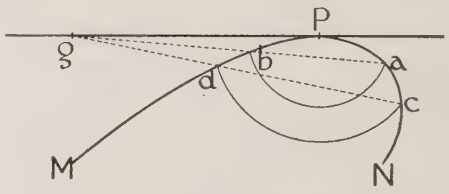


Fig. 2.

c and d. In the given figures the distances Pa, Pb, Pc, and Pd are made greater than is advisable, so as to obtain clearness in the diagram; nevertheless, the direction of the tangent is so very slightly affected by any reasonable modification in the positions of the chords that distances sufficiently great for clear construction may always be taken. In Fig. 3, the point of intersection of the chords is inaccessible and the direction of the tangent is determined by drawing the similar triangles Prs, $p^1r^1s^1$, with corresponding sides parallel and corresponding angular points in the same line.

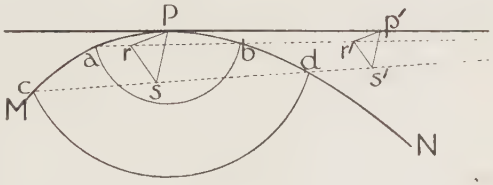


FIG. 3.

Centre and Radius of Curvature.—The radius of curvature at any given point on a curve is the radius of the circle whose curvature most nearly resembles that of the curve at the given point. For the more important mathematical curves the radius of curvature at any point may be found accurately by a definite construction, but where this is not possible the centre of curvature may be found approximately by the method shown in Fig. 4.

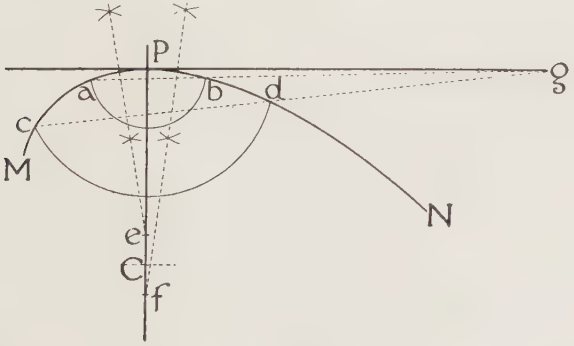


FIG. 4.

Determine the tangent Pg as explained in Fig. 2. Draw the normal PC perpendicular to Pg, and the perpendicular bisectors of Pa and Pb to meet it in e and f. Bisect ef in point C; then C is the centre and CP the radius of curvature of the curve at the point P. For, e is the centre of a circle whose curvature closely resembles that of the arc aP, since both pass through the points a and P and are tangent

to each other at P, and f is the centre of a circle similarly related to the arc bP; but the curvature at P is less than that of the arc aP and greater than that of the arc bP; therefore, since the lengths of the arcs aP, bP are nearly equal, the required centre is somewhere near the mid point of ef, and C is the centre and CP the radius of curvature of the curve at P (approx.).

Rectification of Circular Arcs.—In geometrical stair-building and other circular work it is frequently necessary to determine a straight line equal in length to a circular arc. Such a line can be determined to any desired degree of approximation, although the ratio of the circumference to the diameter of a circle cannot be exactly stated. The value 3·1416, which is usually taken to represent the ratio of circumference to diameter, although approximate, is sufficiently accurate for all practical purposes.

In the workshop, graphical methods are preferred for problems of this type, because the fractions resulting from calculation are seldom of a nature to be measured by an ordinary rule. Moreover, graphical methods are usually quicker and more convenient.

Problem 1. *Fig.* 5.—**To draw a circle having a diameter equal in length to the length of a given circular arc.**

Let AB be the given arc, radius AO. Draw a tangent to the arc at A. Produce AO to o, making Oo equal to half AO. Draw a normal to bisect the arc, and with o as centre, oA as radius, cut the normal in c. Draw a straight line from c to cut the tangent perpendicularly in d. Then d is the centre and dA the radius of the required circle, and the diameter Ab equals the length of the arc AB (approx.). The complete circle would be required only when the length of the arc is to be projected in several directions.

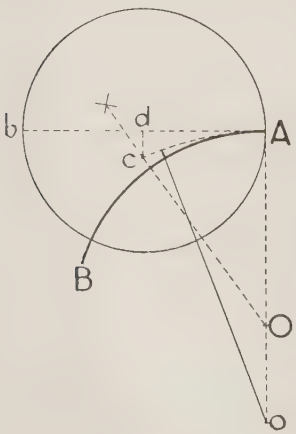


Fig. 5.

Problem 2. *Fig.* 6.—**Develop a circular arc containing any number of given points into a straight line such that the length of the line is equal to the length of the arc, and the distances between the points in the line are equal to the distances between the corresponding points in the arc. Or, given a straight line containing any number of given points, to turn the points into a given circular arc so as to have the distances between the points**

in the arc equal to the distances between the corresponding points in the line.

Let AB be the given arc, radius OA. Produce AO to o, making Oo equal to half AO, and with o as centre, radius oA, draw the arc $A1$; 1 being in the normal bisector of AB. With 1 as centre, and radius $1B$, cut the tangent at A in b. Then Ab equals the length of the arc AB. Let C, D, E, and F be any given points in the arc. Draw the normal bisectors of AC, AD, AE, and AF to cut the arc, centre o, in 2, 3, 4, and 5 respectively. Then 2, 3, 4, and 5 are the centres of the arcs Cc, Dd, Ee, and Ff respectively, and Ac equals the developed length AC, Ad equals the developed length of AD, and similarly for the remaining points.

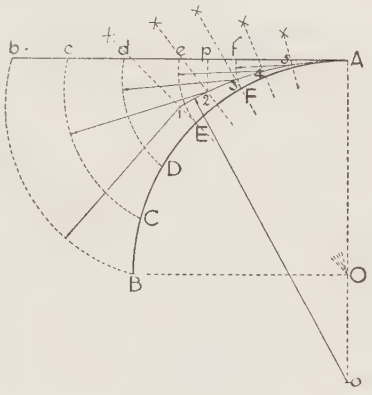


Fig. 6.

The reverse process of transferring given points in the tangent to the given arc is effected by drawing the arc $1A$, centre o, and drawing the perpendicular bisectors of the distances of the points from A. Thus: $p2$ is the perpendicular bisector of Ac, and given Ac, the arc cC may be drawn; and similarly for any other given point in the tangent.

The method explained in this and the preceding problem is not recommended for arcs greater than a quarter circle.

Note.—The method devised by the author will be found to be extremely helpful in the development of stair strings, masonry arches, etc., and is shown applied to the latter in Figs. 262 and 267.

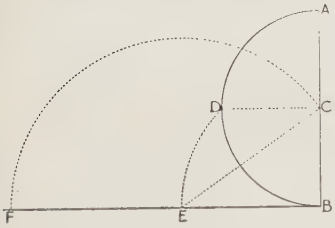


Fig. 7.

Problem 3. *Fig.* 7.—**To determine a straight line equal in length to the length of a given semicircle.**

Let ADB be the given semicircle, centre C. Draw CD perpendicular to AB and draw BF tangent at B. With B as centre, BD as radius, cut the tangent in E. With E as centre, radius EC, cut the tangent in F. Then BF is the required line.

Note.—BF exceeds the length of the semicircle by an amount equal to about $\frac{1}{250}$ part of the radius, but in obtaining the stretch-

out of a semicircular well hole, so frequently required in stair-building, it is better to have the result too great than too small, and since the stretch-out (so called in practice) can be obtained without any additional lines being drawn, the method will meet the requirements of the workshop. The explanatory arcs and lines shown in the figure may be dispensed with if the dividers are used as follows: With one point of the dividers stationed at B, take the distance BD and turn to E, then with the point at E kept in position, take the distance EC and turn about E to F.

Division of Arcs and Angles.—There is no known graphical method of dividing a circular arc or an angle into three, five, six, seven, and so on equal parts, but the following method, devised by the author, will effect such division within the limits of error allowable in mechanical drawing. At the same time it will not involve any elaborate construction, the only lines required to be drawn being the chord and height of the arc, or, in the case of an angle, a subtending chord and arc. In the accompanying figures, the construction lines are shown for the purpose of explanation. The general principle of the method is shown in Fig. 8.

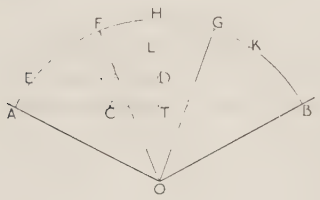


Fig. 8.

Let AOB be any angle, and AHB any arc with centre O, chord AB, and bisector OH. Let C be any point in the chord AB, and DE, LK parallels to AB, such that $TD : TH = AC : AB$ and $TL : TH = BC : BA$. Let F be the point of intersection of circular arcs, centres E and K, radii EC and KC; then arc AF : arc $AB = AC : AB$. It can be shown, however, that the point F is a little above the arc in nearly every case. Thus, suppose the straight line through E and K to be produced to meet BA produced in Q and a parallel to AB through H in R as in Fig. 9: then, if ME equals one third and NK equals two thirds of TH, $QE = KR = EK$ and angle KRH = angle AQE = angle FQE, when QF is a tangent to the arc. But the point of contact of the tangent is S, QS being equal to RH, and S will be in F only when the arc is to be bisected; therefore the point F of the isosceles triangle QCF is in the tangent and not in the arc AB. By an algebraic analysis, too lengthy to be given here, the error in trisecting an angle of 120°

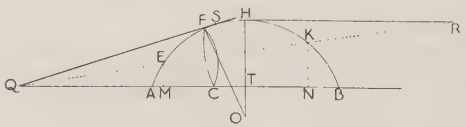


Fig. 9.

is somewhere in the neighbourhood of $\frac{1}{1000}$ when $EF = EC$, F being in the arc.

When the angle is acute, as in Fig. 10, it involves less work if the perpendicular to one arm of the angle is used instead of the chord and bisector of the arc, as follows :

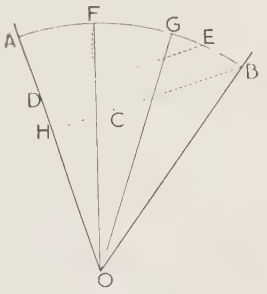


Fig. 10.

To trisect the angle AOB, Fig. 10. Draw HB perpendicular to AO, and make BC equal twice HC. Then, since it is required that $HD : DA = HC : CB$, no further lines are required to be drawn, for CD is parallel to AB, DE is parallel to HB, and E may be located by set squares. Make $EF = EC$, and the angle is trisected by a straight line through O and F (approx.).

Problem 4. *Fig.* 11. —**To divide a given arc or a given angle into five equal parts.**

Let the arc AHB, centre O, or the angle AOB be given. Make $AC : CB = TD : DH = 2 : 3$. Draw DE parallel to AB. With E as centre, EC as radius, cut the arc in F ; then arc AF is two fifths of arc FB, and FH is one tenth of the whole arc AHB. In the figure, HG is made equal to HF, therefore angle GOF is one fifth of the angle AOB. The second point, between F and A, may be determined in a similar manner by making AC equal $\frac{AB}{5}$ and TD equal $\frac{TH}{5}$, but usually it would be quicker to determine G as explained, and step the distance FG along the arc.

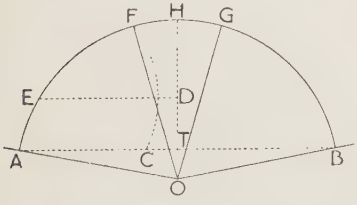


Fig. 11.

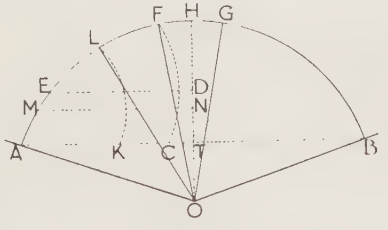


Fig. 12.

Problem 5. *Fig.* 12.—**To divide a given arc or a given angle into seven equal parts.**

Let AOB be the given angle and AHB the given arc. Draw the chord AB, and draw OH perpendicular to it. Make AC equal to $\frac{3}{7}$ AB and TD equal $\frac{3}{7}$ HT. Draw DE parallel to AB. With E as

centre, EC as radius, cut the given arc in F; then arc $AF = \frac{3}{7}$ arc AB. Make HG equal HF; then angle $FOG = \frac{1}{7}$ angle AOB. FG may now be stepped along the arc, or a second point determined by the method used to find point F, thus:

Make AK equal $\frac{2}{7}$ AB and TN equal $\frac{2}{7}$ TH. Draw NM parallel to AB and make ML equal MK; then arc AL equals $\frac{2}{7}$ arc AHB and the angle LOF equals the angle GOF. It should be noted that FH is equal to $\frac{1}{14}$ of AHB.

If a given angle is too small to allow clear working it may be doubled and dealt with as in Fig. 10.

CHAPTER II

TRIANGLES AND QUADRILATERALS

Theorem.—If any triangle be turned about one of its angular points, the triangle moving always in the same plane and the point having a fixed position in that plane, while a second angular point moves along any fixed line, the sides of the triangle increasing or decreasing proportionately, the path of the third angular point is a line similar to the path of the second point, and the angle between the corresponding parts of the two paths is equal to the angle of the triangle at the point of rotation.

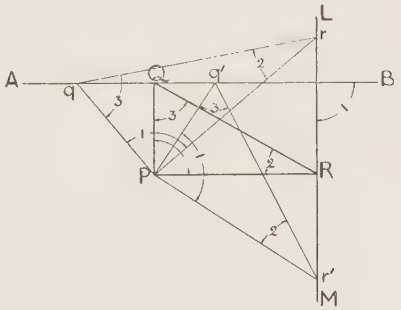


Fig. 13.

Illustration 1. *Fig.* 13.—Let the right-angled triangle PQR turn about the fixed point P, and the sides increase proportionately so that Q moves always in the straight line AB; then the straight line LM will be the locus of R, and, because the angle at P is a right angle, LM will be perpendicular to AB. Pqr, Pq^1r^1 are two positions of the moving triangle.

Illustration 2. *Fig.* 14.—The scalene triangle PQR turning about P, and having Q always in AB, will have R always in LM, and the angle between AB and LM will be equal to the angle RPQ. Pqr, Pq^1r^1 are positions of the moving triangle, the corresponding angles being indicated.

Consider the triangles Pqq^1, Prr^1. Since Pq and Pr have been turned through the same angle, and Pq^1, Pr^1 have also been turned through the same angle, and since $Pq : Pq^1 = Pr : Pr^1$, the triangles Pqq^1, Prr^1 are similar. Also $PQ : Pq = PR : Pr$, and the angles QPq, RPr are equal. Then the two figures $PQqq^1$, $PRrr^1$ are similar in

every respect, and because q, Q, q^1 are collinear, r, R, r^1 are also collinear, and since any other positions of the triangle may be similarly treated, the locus of R is a straight line.

Next consider the position of the triangle, first when R is at the point s, and secondly when Q is at s, s being the point of intersection of AB and LM. When R is at s, the side QR is in AB, and when Q is at s, the side QR is in LM. In moving from the first to the second position, the sides of the triangle have moved through an angle equal to QPR, and QR has moved from a position in AB to a position in LM; then the angle between AB and LM is equal to the angle QPR.

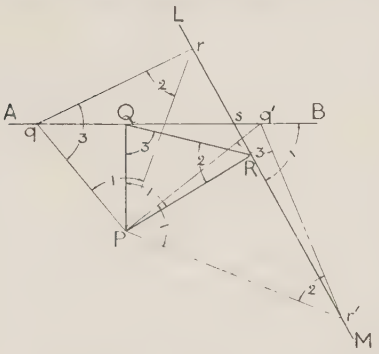


Fig. 14.

Problem 6. *Fig.* 15.—**To draw a triangle *PQR*, similar to a given triangle *abc*, with one angular point on a given straight line, a second on a given curve, and the third in a given point.**

Let AB be the given straight line, P the given point, and CD the given curve.

Draw a line from P parallel to ab, intersecting AB in q. From P and q draw lines parallel to ac and bc, intersecting in point r. Through

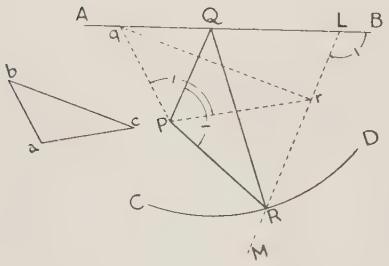


Fig. 15.

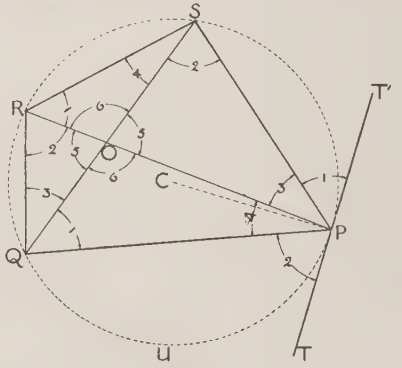


Fig. 16.

r draw the straight line LM, making the angle MLB = angle rPq and cutting the curve in R. Join PR. From P draw a straight line intersecting AB in Q and making the angle rPq with PR. Join RQ. PQR is the required triangle. Compare with Figs. 13 and 14.

In connection with the next problem it is necessary to consider one or two angular properties of the circle.

Theorem.—If from the point of contact of any tangent to a circle any chord be drawn, the angle between the chord and the tangent is equal to the angle in the opposite segment. The angle in a segment is constant.

Illustration. Fig. 16.—TT^1 is a tangent to the circle, centre C, and P is the point of contact. PQ and PS are chords from P. Then the angles SQP, SRP, SPT are equal, and the angles QSP, QRP, QPT are equal.

It should be noted that the angle in a semicircle is a right angle, in a segment greater than a semicircle, an acute angle, in less than a semicircle, an obtuse angle.

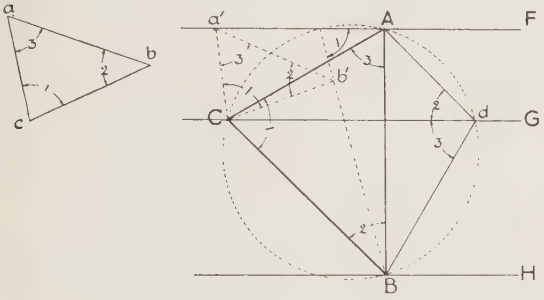


Fig. 17.

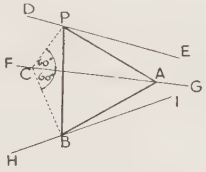


Fig. 18.

Theorem.—In any quadrilateral inscribed in a circle and divided by its diagonals into four triangles, the opposite triangles are similar.

Illustration. Fig. 16.—First consider the opposite triangles QOP and ROS. The angles OQP, ORS are equal, both being angles in the same segment on SP. Similarly, the angles OPQ, OSR are in the same segment on QR, and are therefore equal. Then the triangles ROS, QOP are similar. By the same process the triangles ROQ, POS can be shown to be similar.

Problem 7. *Fig.* 17.—**To draw a triangle similar to a given triangle, with its angular points one in each of three given straight lines.**

Solution 1.—Draw the triangle a^1Cb^1 similar to the given triangle abc, with one angular point in the given line F, and a second angular point in the given line G. Draw b^1B to make an angle with the line F equal to the angle a^1Cb^1, cutting the line H in B. Then BC is the side of the required triangle corresponding to bc.

Solution 2.—Referring again to Fig. 16, it will be seen that the opposite angles of a quadrilateral inscribed in a circle are supple-

mentary ; for the triangle QRS contains the angles $1+2+3+4=180°$. The angle QRS equals angles $1+2$. The opposite angle QPS equals angles $3+4$; therefore the angles QRS, QPS are together equal to two right angles. In the same way the angles at S and Q can be shown to be supplementary.

In Fig. 17 this property is made use of for a second method of solution, thus : If at any point d, in the line G, Ad and Bd be drawn, making the angles AdC, abc equal, and BdC, bac equal, then a circle through AdB will cut the line G in C and ABC is the required triangle. Compare with Fig. 16.

Fig. 18 shows the same method applied in placing an equilateral triangle with its angular points one in each of three given straight lines, and should require no explanation.

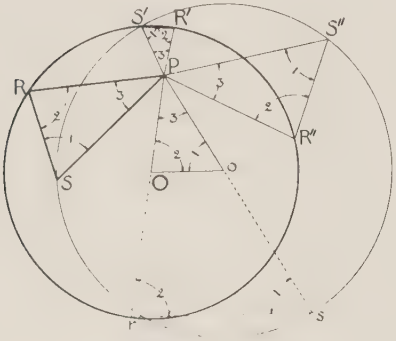


Fig. 19.

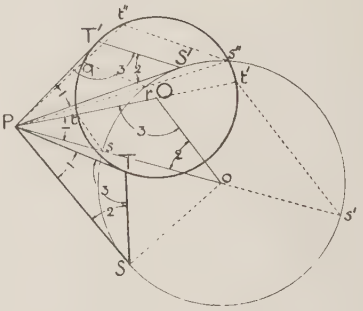


Fig. 20.

Theorem. If any triangle turns about one of its angular points while a second angular point travels in the circumference of a plane circle, the sides of the triangle increasing or decreasing proportionally, then the third angular point describes a second circle, whose centre is in one of the angular points of a triangle similar to the moving triangle, one side of which is the line joining the centre of the former circle to the point of rotation ; the position of the triangle corresponding to one position of the moving triangle. The ratio of the radii of the two circles is equal to the ratio between the two sides of the triangle which meet at the point of rotation.

Illustration. Fig. 19. — Let the triangle PRS turn about P, while R travels around the circumference of the circle, centre O, the sides of the triangle increasing and decreasing proportionally ; then S will be always on the circumference of the circle, centre o, and the triangle POo will be similar to the triangle PRS. Four positions of the moving triangle are shown, and corresponding angles are indicated.

It will be seen that the radius Or is to the radius os as PR is to PS.

When S reaches a point of intersection of the two circles, both S and R are in the same circle, centre O, one such position being shown at PR^1S^1.

Fig. 20 is a second illustration of the theorem, the point of rotation being, in this case, outside the circle. PTS is the triangle turning about P, T being always in the circle, centre O. Then the locus of S is the circle, centre o, and the triangle POo is similar to PTS.

When PT is a tangent to the path of T, PS is a tangent to the path of S, and, for other positions of the triangle, the circles cut the sides PT, PS proportionally ; thus, $Pt : Pt^1 = Ps : Ps^1$.

Further illustrations of the relationship between the loci of the two

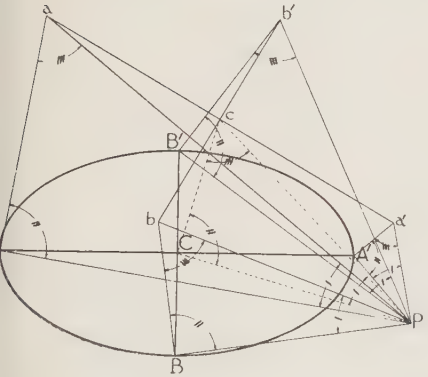


Fig. 21.

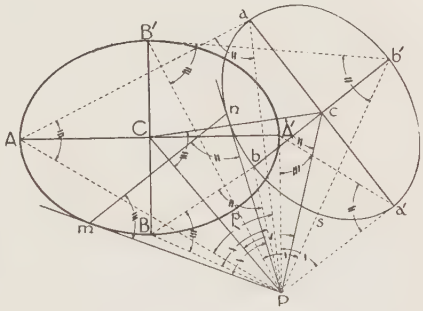


Fig. 22.

moving angular points in a triangle turning about the other angular point are given in Figs. 21, 22, 23.

In Figs. 21 and 22 P is the point of rotation, AA^1, BB^1 the axes of the elliptic path of the second point, and aa^1, bb^1 the axes of the elliptic path of the third point of a triangle always similar to PAa. In Fig. 21 the triangle is right-angled, and it will be seen that a, a^1, b, b^1, and c are in the corresponding angular points of five similar triangles, and five corresponding sides of these triangles connect, Aa, Bb, A^1a^1, B^1b^1, Cc, each of the triangles having its third angular point in P.

In Fig. 22 both elliptic curves are drawn and the two tangential positions of the triangle are shown at Pmn and PA^1a^1 ; m, n, A^1, a^1 being the points of contact. The ratio of corresponding lines in the two ellipses is equal to the ratio of the two sides of the triangle which meet at P. Therefore, $aa^1 : AA^1 = Pa : PA = bb^1 : BB^1 = B^1r : b^1s$.

In Fig. 23 the loci of the two moving points are parabolas. P is the point of rotation. The triangles PFf, PAa, PDd, PRr, PSs, and PQq are similar. F is the focus, A the vertex, and D is in the directrix.

If the previous figures have been studied, no further explanation is necessary.

Problem 8. *Fig.* 24. —**The distances of three equidistant points *A*, *B*, and *C* from a fourth point *O* are given. Determine the distance between the points *A*, *B*, *C*. Let *OA*, *OB*, and *OC* be the given distances.**

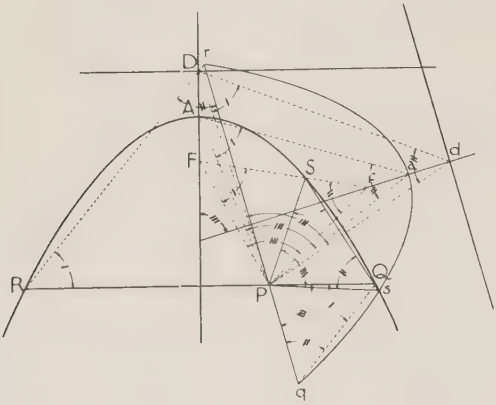


Fig. 23.

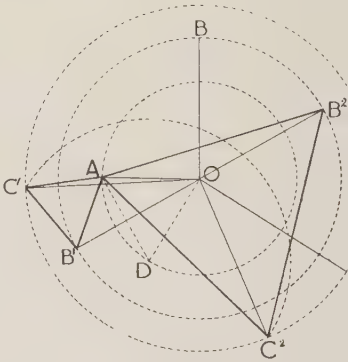


Fig. 24.

With centre O, radii OA, OB, and OC, draw circles through A, B, and C. On OA erect the equilateral triangle AOD, and with D as centre, radius OB, describe a circle cutting the circle through C in C^1 and C^2. Then AC^1 and AC^2 are respectively one side of two equilateral triangles that have an angular point in each of the three circles. The nature of the problem will decide which is the one required.

CHAPTER III

THE CIRCLE

Theorem.—The angle subtended by any chord at the centre of a circle is equal to twice the angle subtended by the same chord at the circumference of the circle.

Problem 9. *Fig.* 25.—**To cut off from a given circle a segment to contain a given angle.**

Let O be the centre of the given circle. Draw any radius OD, and draw OB, making the given angle with it. Draw BA perpendicular to OD; then, if the given angle is less than 90°, the segment AEB containing the centre of the circle is the one required; if the given angle is greater than 90°, the smaller segment ADB contains it.

Equal angles in the figure are indicated by the same numbers.

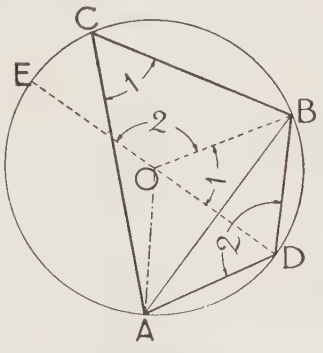


Fig. 25.

Note.—The angle DOB is half the obtuse angle AOB, and the angle EOB is half the reflex angle AOB. Also, 90° − angle ABO = angle ACB; and 90° + angle ABO = angle ADB.

Problem 10 (*no figure*).—**To draw a segment on a given line to contain a given angle.**

Let the given line be AB. From A or B draw a line making with the given line an angle equal to 90° − the given angle, or the given angle − 90°, according as the angle is greater or less than a right angle. The point of intersection between this line and the perpendicular bisector of the given line is the centre from which to draw the required arc.

Other solutions to problems 9 and 10, based on the preceding theorem, should be apparent to the student who has considered

Fig. 25, and where there are no objections to a tangent being drawn, might be preferred.

Theorem.—If any chord of a circle be extended to intersect a tangent to the circle, the distance between the point of intersection and the point of contact of the tangent is the mean proportional, or geometrical mean, between the distances of the point of intersection from the extremities of the chord.

Illustration. Fig. 26.—Let the chord SR be extended to intersect the tangent at T in P; then $PT^2 = PR \times PS$.

Theorem.—If chords be drawn through any fixed point within a circle, then the product of the segments of the chords is constant.

Illustration. Fig. 26.—Let FG and KH be any chords through

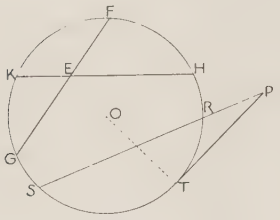


Fig. 26.

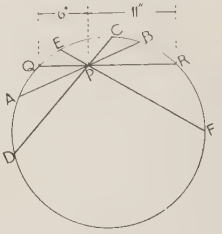


Fig. 27.

the fixed point E; then, GE multiplied by FE equals KE multiplied by HE.

Practical Application. Fig. 27.—In stair-designing it is desirable to make the width of the tread in inches, multiplied by the height of the riser in inches, equal 66. Since the riser is usually obtained by dividing the perpendicular distance between the floors into a suitable number of equal parts, the height of a riser may be such as to introduce a very inconvenient fraction into a numerical calculation, and the following graphical method of determining a suitable tread would then be desirable.

In any suitably large circle draw the chord QR equal to 17 inches and mark point P 6 inches from Q; then $QP \times RP = 66$, and the segments of any chord through P will give the same product. Take the height of the proposed riser in the compasses, and with P as centre cut the circumference of the circle in a point, such as E, C, or B. Then for riser = to PE, the tread equals PF; riser = to PC, the tread = PD; riser = to PB, the tread = PA.

Radical Axis.—The radical axis of any two circles is a straight line

perpendicular to the line joining the centres of the circles and such that tangents drawn to the two circles from any point in the line are equal.

Illustration. *Fig.* 28.—O, O^1 are the centres of the given circles. Any circular arc is drawn to cut the two circles in four points, r, s, t, u, or m, n, p, q. Then the common chords rs, tu produced to meet in w,

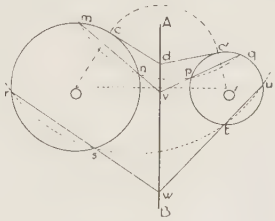


Fig. 28.

or mn, pq produced to meet in v, will give a point in the radical axis AB, and AB is always perpendicular to OO^1. When the circles touch, AB passes through the point of contact. When the circles cut each other, AB passes through the points of intersection.

In the given figures dc, dc^1 are tangents intersecting on AB, then $dc = dc^1$. This may be proved by the theorem illustrated in Fig. 26.

Graphical method of finding the mean proportional between two given straight lines AB and BC.

Fig. 29.—Let AB and BC be the given lines arranged in the same straight line. Draw a semicircle on AC and draw BD perpendicular to AC; then BD is the required mean.

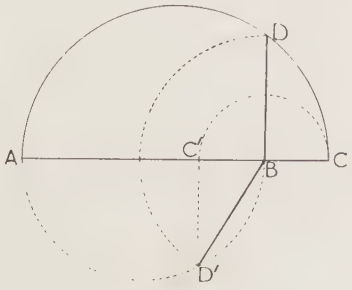


Fig. 29.

A second method is shown in dotted lines. Make BC^1 equal to BC; draw a semicircle on AB, and the perpendicular C^1D^1. Join BD^1. Then $BD^1 = BD$.

Centres of Similitude. *Fig.* 30.—Let O, o be the centres of any two circles, and RQ, rq any two parallel diameters; then straight lines through Rr, Qq intersect in P and straight lines through Rq, rQ intersect in p. P, p are the centres of similitude of the two circles and are always in the straight line through Oo. TS, ts, common tangents to the two circles, produced if necessary, pass through P or p.

It should be noted that the triangles PRQ, Prq are similar; also pRQ, prq are similar, and therefore $Po : PO = or : OR$, and $po : pO = or : OR$.

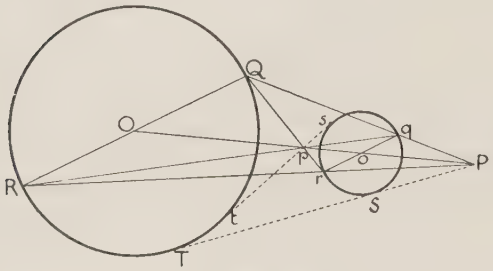


Fig. 30.

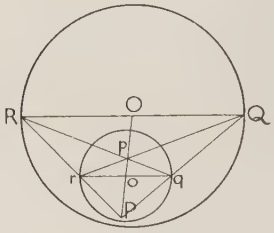


Fig. 30*a*.

When two circles touch, one centre of similitude is in the point of contact. Fig. 30*a* shows the positions of the centres of similitude when one circle contains the other.

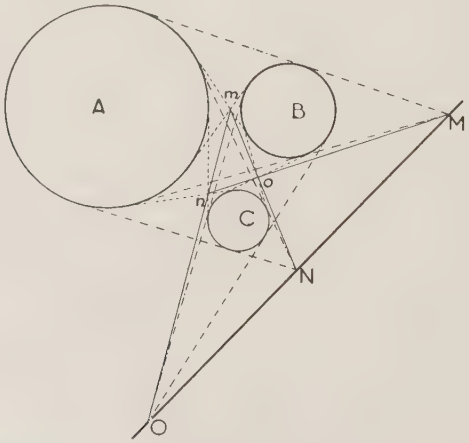


Fig. 31.

Axis of Similitude. *Fig.* 31.—For any three circles, taken two at a time, there are six centres of similitude. Let the three that occur between the circles be referred to as the inside centres, and three

beyond the circles as the outside centres; then the three outside centres lie on one straight line, which is called an Axis of Similitude. In the figure, the centres of similitude for circles A and B are M, m; for circles A and C, N, n; and for circles B and C, O, o.

The inside centres are not in one straight line, but any two lie in a straight line which passes through the outside centre of the other two circles; thus, mnO, moN, noM, MNO are in each case collinear.

Example of the Use of the Axis of Similitude.

Problem 11. *Fig.* 32.—**Given the straight lines *PG* and *RS* and the circle centre *O* projecting between the lines, to draw the two circles to touch *PG* and *RS* and the given circle.**

Draw any two circles, centres M and N, to touch the given lines. Determine A and B, the inside centres of similitude of the given circle and the circles M and N. A straight line through

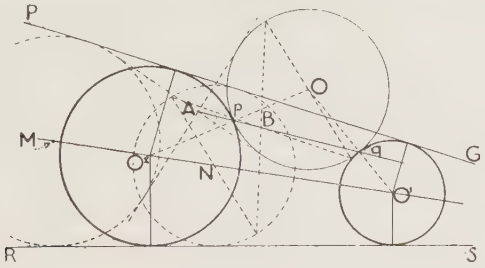


Fig. 32.

A and B will cut the given circle in points p and q; then p and q are the points of contact between the given circle and the required circles, the centres of the latter being O^1 and O^2. If the point of intersection of the given lines PG, RS were accessible, only one of the construction circles, M, N, would be required, since the line through A and B would pass through the point of intersection, this point being the outside centre of similitude of any two circles drawn to touch PG and RS.

CHAPTER IV

PROBLEMS ON TANGENTIAL LINES AND CIRCLES

Problem 12. *Fig.* 33.—**Given a straight line *AB* containing a fixed point *P*, and a circle, centre *O*, to draw the two circles to touch *AB* at *P*, and touch the given circle.**

Draw a perpendicular to AB at P. With any suitable radius draw a circular arc to touch AB at P, and cut the given circle in any two points D and E. Through ED draw a straight line to intersect AB in C; then CP is the length of tangents from C to any circle passing through D and E, and of course to any circle touching AB at P. With C as centre, CP as radius, cut the given circle in M and N;

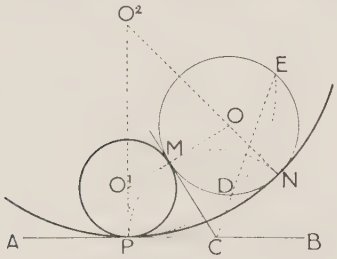


Fig. 33.

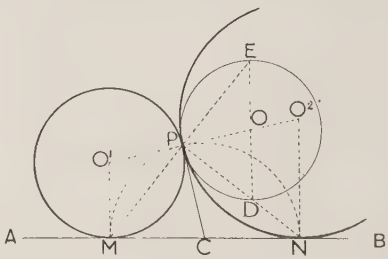


Fig. 34.

draw straight lines through OM and ON to cut the perpendicular from P in O^1 and O^2. O^1, O^2 are the centres of the required circles.

Problem 13. *Fig.* 34.—**To draw circles to touch a given circle at a given point, and touch a given straight line. Let *O* be the centre of the given circle; *P* the given point; and *AB* the given line.**

Draw a tangent to the given circle at P, to cut AB in C. Make CM and CN equal to CP. A line through OP to intersect perpendiculars to AB from points M and N gives O^1 and O^2, the centres of the required circles.

Problems 12 and 13 may be solved by a second method, as follows:

Consider the points of contact between the circles as centres of similitude; then, by drawing a diameter of the given circle perpendicular

to AB, and drawing straight lines from P through its extremities, M and N are determined. In Fig. 34 the diameter of the given circle perpendicular to AB meets the circumference at E and D; then, straight lines drawn through EP and PD give M and N.

The method just explained is applied to a problem similar to problem 13 in Fig. 35. Here the given circle, centre O, is cut by the given straight line AB, and again the required circles are to touch AB at the given point P, and touch the given circle.

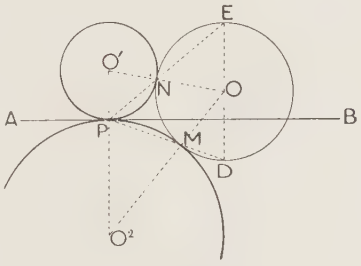


Fig. 35.

Draw the diameter ED perpendicular to AB and join EP and DP; the points M and N, where EP, DP cut the given circle, are the points of contact between the given and the required circles. Note that the triangles NO^1P, NOE are similar and isosceles, as are also MOD, MO^2P.

Problem 14. *Fig.* 36.—**To draw circles to pass through two given points and touch a given straight line.**

R and S are the given points and AB is the given straight line. Draw any circle through R and S to cut AB in points E and F. Draw a straight line through R and S to intersect AB in C. Draw a semicircle on CF and drop a perpendicular to AB from E to cut it in D; RS and EF being chords of the same circle and C their point of intersection, then CD is the length of a tangent from C to any circle through R and S. Make CN, CM each equal to CD, and from M and N erect perpendiculars to meet the perpendicular bisector of RS in O and O^1. O^1N, OM are radii of the required circles.

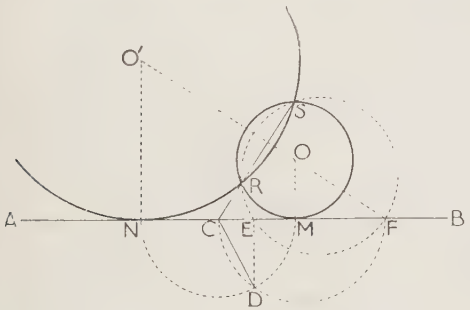


Fig. 36.

Problem 15. *Figs.* 37 *and* 38.—**To draw circles to pass through two given points and touch a given circle.**

In Fig. 37 the given points R and S are inside, in Fig. 38 outside, the given circle, centre O. Through R and S draw any circle to cut

the given circle in A and B. Through SR and AB draw straight lines to intersect in P. Tangents from P to the given circle are tangents to the required circle, since AB is a chord of the given circle and also

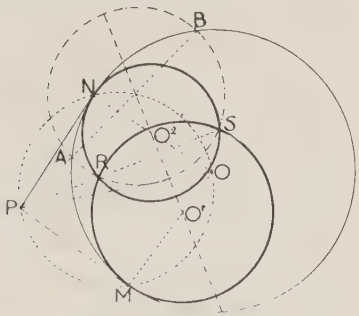


Fig. 37.

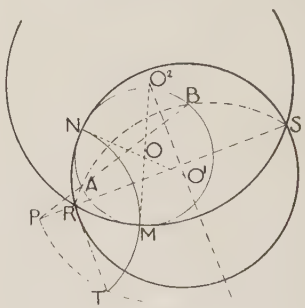


Fig. 38.

of a circle through RS; and M and N are the points of contact between the tangents and the given circle and of the two required circles. Perpendiculars to PN and PM intersect the perpendicular bisector of RS in O^1 and O^2, the centres of the required circles. In Fig. 37 M and N are found by describing a circle on PO; in Fig. 38 by finding RT, the mean proportional between PR, PS, and making PM, PN equal to it. The former involves less drawing, but the latter better illustrates the geometric principle upon which the solution is based.

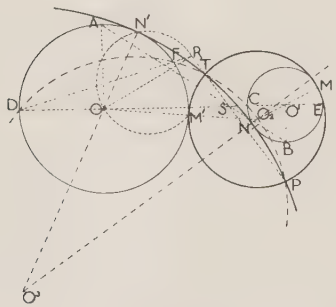


Fig. 39.

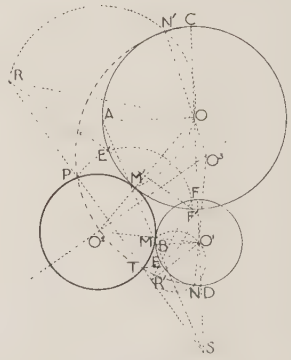


Fig. 40.

Problem 16. *Figs.* 39 *and* 40.—**To draw circles to pass through a given point and touch two given circles.**

Consider Fig. 39, where the required circle is to have internal contact with one given circle and external contact with the other.

Let P be the given point and OAD, O^1CB the given circles. Draw

the arc of a circle to pass through D, C, and P. Find S, the inside centre of similitude, and draw a straight line through PS to cut the arc DCP in a point T. Circles drawn through P and T to touch one given circle will touch the other circle. The arc DCP cuts the circle, centre O, in a second point F. Draw straight lines through P, T and D, F to intersect in R. Determine M^1, N^1, the points of contact of tangents from R. Straight lines through N^1O and OM^1 intersect the perpendicular bisector of PT in O^2 and O^3, the centres of the required circles.

Consider Fig. 40, where one of the required circles is to have internal contact, the other external contact with both the given circles.

Through C and D, the remotest points in the given circles, and the given point P draw a circular arc. Find S, the outside centre of similitude. Draw PS to cut the arc in T. Circles through P and T, drawn to touch one given circle, touch the other also. Since the points of intersection of the arc DPC with either of the given circles are comparatively near to one another, a more satisfactory construction than that given in the previous example is as follows :

Draw any circle through P and T to cut the given circles in E, F, and E^1, F^1. Through E, F, and E^1, F^1 draw straight lines to meet the line through S, P, in R and R^1. The four points of contact M, N, M^1, N^1 may now be determined by drawing circles with RO and R^1O^1 as diameters. The centres of the required circles O^2 and O^3 can be found as before. Only one circle is drawn in the figure.

Problem 17. *Fig.* 41.—**To draw a circle to touch three given circles.**

Where the required circle is to have internal contact with the three given circles, reduce each of the circles by an amount equal to the radius of the smallest circle. The problem is now the same as the preceding one.

Only one of the required circles is shown in the figure, but by the process of enlarging and reducing the radii of the given circles it is sometimes possible to find eight solutions.

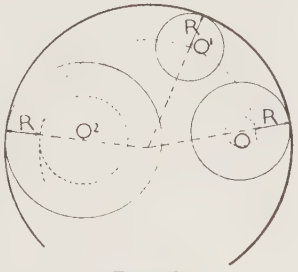


Fig. 41.

Problem 18. *Fig.* 42.—**To draw circles to touch a given circle and a given straight line and to pass through a given point.**

O is the centre of the given circle, P is the given point, and QR is the given line.

Draw the straight line $AOBD$ perpendicular to QR. Now regard

the line QR as part of the circumference of a circle of infinitely large radius, its centre on the side away from O; then B is the centre of similitude of this and the given circle. If the imaginary centre and centre O are on the same side of QR, then A is the centre of similitude. (Refer back to Figs. 30 and 30*a* and the section on centres of similitude, if the foregoing is not fully understood.)

A and D are also the corresponding ends of diameters, in the same straight line, of the given circle and the imaginary circle. Considered in this way, the construction is the same as in Figs. 39 and 40, therefore a straight line through P, B cuts a circle through A, D, P in a point C; and a circle through C and P tangent to QR will touch the given circle. The points of contact p and c are determined, in the given example, by extending the line through P and C to E, a point in the common chord of the two circles through A.

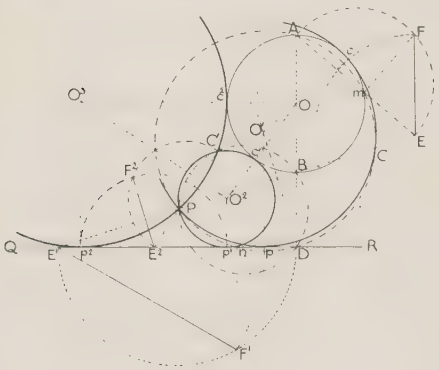


Fig. 42.

produced, and to E^1, in QR, and constructing semicircles on EA and E^1D to cut perpendiculars to EA from m, and to E^1D from n, in F and F^1; $Ec = EF$ and $E^1p = E^1F^1$. A line through cO meets a perpendicular to QR, from p, in O^1, the centre of one required circle.

It should be noted that the arc Fc, if continued, would cut the given circle in a second point, and that E^1F^1 may be turned to cut RQ produced on the other side of E^1; thus the points of contact of a second circle are obtained. This circle is too large to be shown.

Two other circles are determined by drawing a straight line through AP to intersect QR in E^2, and a circle through B, D, P in C^1. The remainder of the construction is shown in the figure and the student would not benefit by further explanation.

Several diagrams, particularly Fig. 42, are full to the point of appearing complex; the author's view that this is an advantage is perhaps unusual, but it is hoped that the bright student will find additional interest, and the less bright, additional help in the apparent complexity.

CHAPTER V

SOME APPLICATIONS OF TANGENTIAL CIRCULAR ARCS

In Fig. 43 four arrangements of the two circular arcs forming the outline of one side of an ogee roof are shown. In each case the line AE is a normal to the lower arc, and A^1F a normal to the upper arc; therefore the points of contact must lie on EF, E and F being the extremities of parallel diameters. Construction: When the two arcs are required to be equal, bisect EF in a and draw the perpendicular bisector of Ea to meet EA in A. A is the centre, AE the radius of the lower arc which cuts EF in a. Through A, a draw a straight line to meet the horizontal line from F in A^1; then A^1 is the centre, and A^1F the radius of the upper arc. NOTE.—A is not necessarily in the foot of the perpendicular from F, but can always be so arranged when the length AE on AF is not fixed.

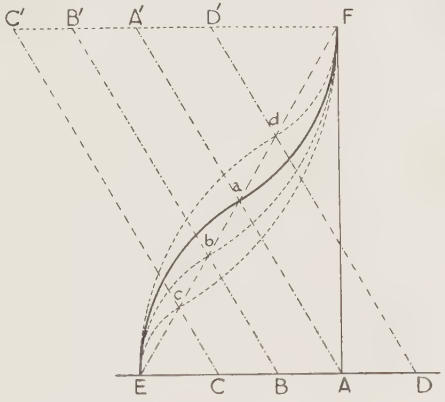


FIG. 43.

When the two arcs are not required to be equal, any centre in either AE or FA^1 may be selected and one arc drawn to cut FE in, say, c; then the other centre may be found as explained.

The proof, which is based on similar isosceles triangles, should not give any trouble.

A dimensioned, six-centred oval is shown in Fig. 44. Construction: Draw the two circles OAe, O^1Bf and the line CD to the given sizes and in the given positions. Make Cg and Cg^1 equal to O^1B and OA respectively. Bisect g^1O, O^1g by perpendiculars intersecting CD in O^2 and CD produced in o^1, the centres of the required arcs Ce^1, Cf^1.

The corresponding centres could now be plotted, but a second method of finding them is shown in the figure. Draw the radii Oh, O^1h^1 and through Dh and Dh^1 draw straight lines to intersect the circles OA and O^1B in points e and f. Then e and f are the points of contact, and straight lines through eO and fO^1, to meet DC in O^3 and DC produced in o, give the required centres for the arcs De and Df.

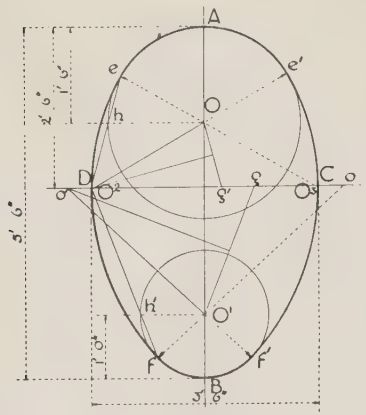


Fig. 44.

The outline of an egg-shaped sewer is an oval similar to Fig. 44 and may be set out in the same way.

In the true semi-elliptical arch, where the joint lines are normal to the curve, no two voussoirs on the same side of the centre line are the same shape. In consequence of this, the amount of work involved in cutting the voussoirs is so great that a curve, made up of three circular arcs and sometimes called a false ellipse, is substituted.

Problem 19. *Fig.* 45.—**To draw the outline of a three-centred arch of given rise and span, closely resembling the true elliptic curve.**

Two examples, one on each side of the centre line, are shown in the figure, each small letter on the right corresponding with the same capital letter on the left. Let AC equal half the given span, CB the given rise.

Construction: Bisect AE in F. Join BF. From A draw AD at 45° to AC intersecting FB in D. Draw a perpendicular to AC from D to intersect AC in O^1. With O^1 as centre, O^1A as radius, draw the arc AP intersecting FB in P. Through PO^1 draw a straight line to intersect BC produced in O; then O, O^1 are two of the required centres; the third centre corresponds with O^1 on the other side of BC.

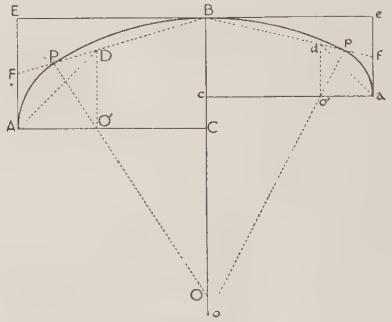


Fig. 45.

Problem 20. *Fig.* 46.—**Given the span, rise, and the radius of one**

of the circular arcs for a three-centred arch, to find the centre and radius of the second arc, and the point of contact of the arcs.

Let AC equal half the given span, BC equal the given rise, and OB the given radius. Draw OA^1 perpendicular and equal to OB. Draw the quadrant BA^1, and through A^1A draw a straight line to cut it in a point P. Draw the straight line PO to cut AC in O^1; then O^1 is the required centre, P the point of contact, and O^1P the required radius. For, OA^1, OP are radii of the same circle, therefore the triangle OA^1P is isosceles; and because AO^1 is parallel to A^1O, AP in A^1P, and O^1P in OP, the triangle O^1AP is isosceles. P is the point of contact, since it is a point in both arcs and is also in the straight line through the two centres. Construction when radius of the smaller arc is given: Let O^1A be the given radius. Draw O^1D perpendicular and equal to O^1A, and draw the quadrant AD. Through B and D draw a straight line to cut the quadrant in a point P. Then P is the point of contact, and O, the required centre, is the point of intersection between PO^1 and BC, produced.

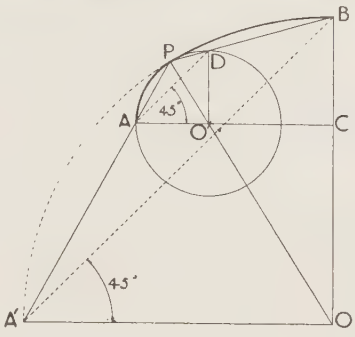


Fig. 46.

Note.—O^1PD, OPB are similar isosceles triangles. AD is drawn to show the connection between Figs. 45 and 46.

SCROLL CURVES

Figs. 47 and 48 illustrate the geometrical principles involved in setting out ornamental scrolls and volutes, made up of tangential arcs of circles, in which several dimensions are given.

In Fig. 47, the curve $ABCDE$ is made up of four tangential quadrants of circles: AB, centre p; BC, centre q; CD, centre r; and DE, centre s. Then for any three radial decreases or centre displacements, pq, qr, rs, and any initial radius pA or sE, $X = pq + 2qr + rs$; X being the distance EA between tangents at A and E.

In Fig. 48 the effect of each radial decrease and centre displacement in producing the value X is shown. Let pa be the initial radius, qb the second radius, rc the third radius, and sd the fourth radius, and let the arcs cb and cd be continued to b^1 and c^1; then it will be seen that when the centre displacement is perpendicular to pa, as pq and

rs, only the radial decrease affects X, and when the centre displacement is parallel to *pa* and away from *a*, as *qr*, X is affected by both the decrease and the displacement. Therefore $ae = y = pq$, $b^1c^1 = x = 2rq$, $fd^1 = z = sr$, and $x + y + z = X$.

When the displacement is parallel to *pa* and towards *a*, X is unaffected.

If the distances between parallel tangents at the extremities of the arcs be denoted by L, M, and N, then $L = 2pa - pq$, $N = 2pa - (2pq + qr)$, and $M = 2pa - (2pq + 2qr + rs)$.

For any given radius PA, and any given distance X, any number of curves may be drawn. In Fig. 49 the two curves are drawn by dividing X into four divisions, first as shown at *a*, *b*, b^1, *c*, and second

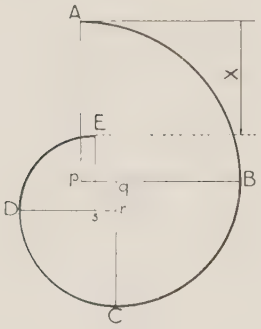


Fig. 47.

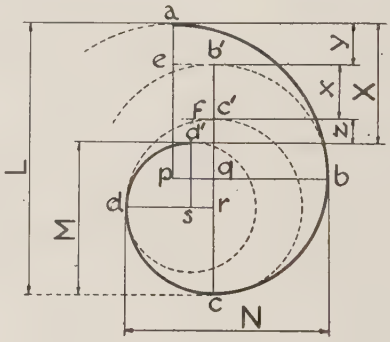


Fig. 48.

at *d*, *e*, e^1, *f*. ($b = b^1$ and $e = e^1$.) Then, by making $pq = d$, $qr = e$, and $rs = f$, the centres for the curve AE are determined.

For the partly dotted curve AE^1, $pq^1 = a$, $q^1r^1 = b$, and $r^1s^1 = c$. Any other method of dividing X into four divisions, two of the divisions being equal, will give other centres.

In Fig. 50 the radial decreases are equal, therefore the centres are in the four corners of a square. In this case X equals the perimeter of the square and is in accordance with the general case. When the smallest radius equals the side of the square, the values L, M, N, and X, as shown in Fig. 48, can readily be verified. It should be evident also, from Fig. 50, that X is constant for all initial radii.

One complete revolution of any spiral curve is called a convolution. In curves of more than one convolution all the convolutions conform to the same law.

Problem 21. *Fig.* 51.—**To draw a scroll curve of two convolutions**

made up of quadrants of circles, the centres to be in the angular points of two squares ; given, the overall width *X*, and the relative positions of the extremities of the two convolutions, *A*, *E* and *I*.

Draw *AK* equal to *X* and mark the points *E* and *I* at the required

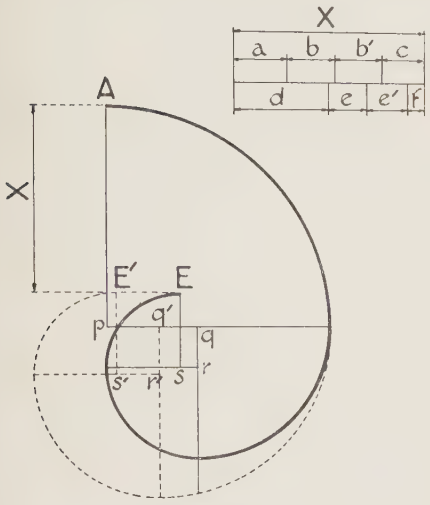


Fig. 49

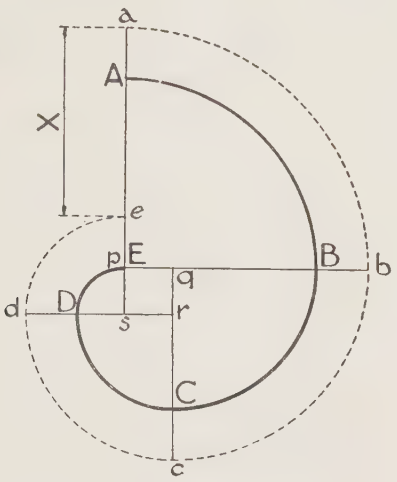


Fig. 50.

distances from *A*. In the given example *AE* equals one fourth and *EI* one eighth of the total width. Bisect *AK* in point *o*, and make *o*1

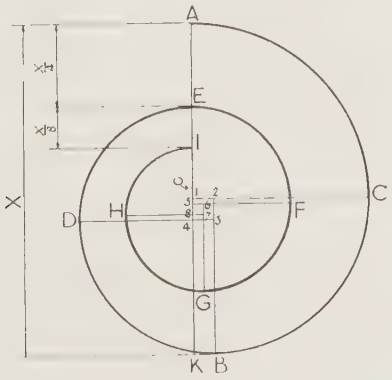


Fig. 51.

equal to one eighth of *AE*. Draw 1*C* perpendicular to *AK* and make 12 equal to one fourth of *AE*. Complete the square 1234 and the

centres for the first convolution *AE* are obtained. For the second convolution make 56 equal to one fourth of *EI* and equidistant from 1 and 4; then the corners of the square 5678 are the required centres.

Problem 22. *Fig.* 52.—**To draw a scroll curve of three convolutions made up of quadrants of circles, the centres to be in the angular points of three squares; given, the overall width, the radius of the circular eye, and the positions of the points *E* and *J* in which the curve intersects the diameter *AP*.**

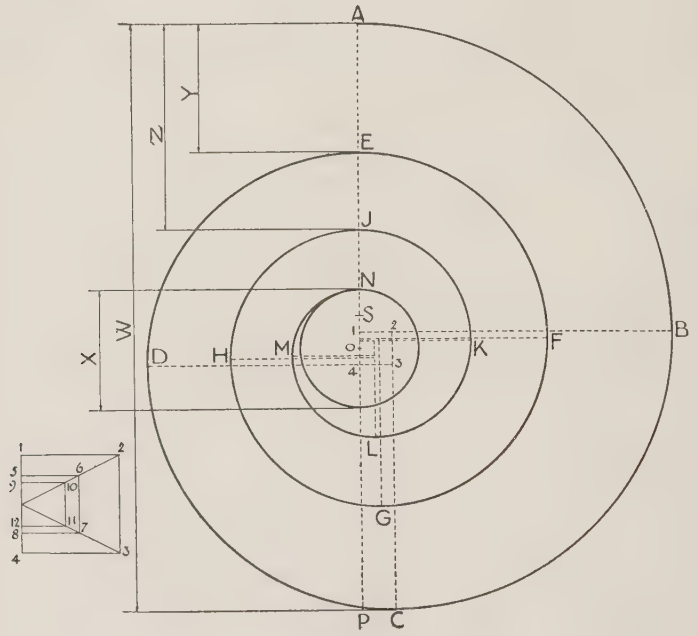


Fig. 52.

Let *W* equal the given width, *X* equal the given diameter of the eye, *Y* and *Z* equal the given distances *AE* and *AJ*.

Draw *AP* equal to *W* and on it mark the points *E* and *J*, the given distances from A. Bisect *AP* in *S* and make *SO* equal to $\frac{Y}{4}$. Bisect *SO* in 1 and draw the square 1234 with sides equal to *SO*; then 1, 2, 3, 4, are the centres required for the first convolution, *ABCDE*.

With centre *O* and the given radius, draw the circular eye, cutting *AO* in *N*. The centres for the three convolutions are shown in the

enlarged detail of the centres. The sides of the square, corresponding to 5, 6, 7, 8, are each equal to $\frac{EJ}{4}$, and the sides of the smallest square, corresponding to 9, 10, 11, 12, are each equal to one fourth of the distance JN; then 5, 6, 7, 8 are the centres for the second convolution, $EFGHJ$, and 9, 10, 11, 12 for the third convolution, $JKLMN$.

Problem 23. *Fig.* 53.—**To draw a volute with graduated centre displacements; given, the overall width, the number of convolutions, the positions of the extremities of the convolutions, and the radius of the circular eye.**

Let W equal the given width XY, and Z equal the distances between the extremities of the convolutions.

Draw AQ equal to W, and perpendicular to it draw Aa, Ee, Kk,

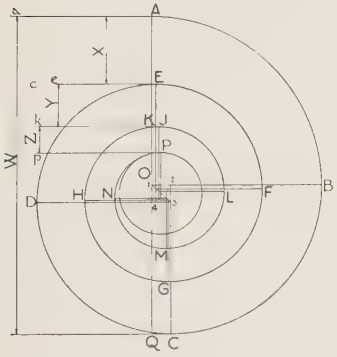


Fig. 53.

Fig. 53*a*.

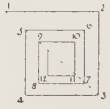


Fig. 53*b*.

and Pp, making the distances $AE = X$, $EK = Y$, and $KP = Z$. Bisect AQ in O, and in any convenient position draw a^1e^1 (Fig. 53*a*) equal to AE. Bisect a^1e^1 in c and bisect ce^1 in d. Select any point o, and draw a^1o, co, and do. Parallel to a^1e^1 draw e^2k^1 and k^2p^1 equal to EK and KP. Draw bo, such that cb is less than a^1b and greater than e^2f; then, referring to the enlarged diagram of the centres, Fig. 53*b*, 1, $2 = a^1b$, 2, $3 = cd$, 3, $4 = cb$. Join 1, 3 and draw 4, 5 parallel to 2, 3, 5 being in the line 1, 3. Make 5, $6 = e^2f$, 6, $7 = gh$, and 7, $8 = fg$. 9 is in the line joining 1, 3. 9, $10 = k^1i$, 10, $11 = mn$, 11, $12 = im$. The point s, the centre of the circular eye, is in the line joining 1, 3. This rather elaborate construction has been devised to meet the requirements of draughtsmen who prefer to sketch first a pleasing volute, and afterward reproduce with the aid of compasses, and for cases where the figure is required to conform to a set of measurements taken from an existing example.

The student should have no difficulty in modifying the construction for cases where fewer fixed distances allow greater latitude.

In handrail scrolls the fixed conditions usually consist of a given overall width, a given width of handrail, and the approximate position of the point at which the inner and outer curves shall meet; the latter point is usually required to be at, or a little beyond, the straight line containing the first two centres, as shown at *e* in Figs. 54 and 55. If the point of intersection *e* is carried much beyond the position shown in Fig. 55, the work involved in moulding the scroll is very much increased and there is little, if any, gain in appearance.

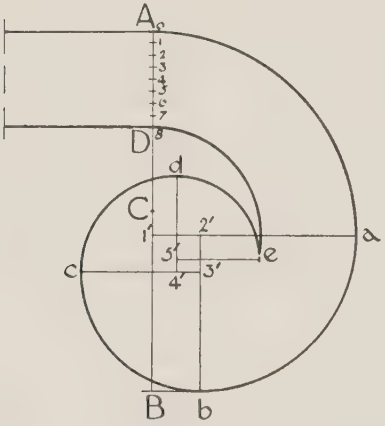


Fig. 54.

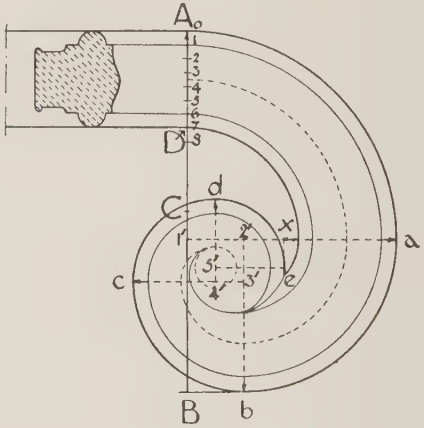


Fig. 55.

Problem 24. *Figs.* 54 *and* 55.—**To draw the outline of a handrail scroll; given, the overall width, the width of the handrail, and the point of intersection of the two curves; the radial decreases to be in arithmetical progression.**

Let AB, Fig. 54, be the given width of the scroll, AD the given width of the handrail, the point of intersection to be at the end of the fifth quadrant of the outside curve.

Divide AD into eight equal parts. Bisect AB in C, and make $C1^1$ equal to two parts of AD. Draw 1^1a perpendicular to AB, and on it set off the distance 1^12^1 equal to four parts of AD. Draw 2^1b parallel to AB and make 2^13^1 equal three parts of AD. Draw 3^1c and 4^1d perpendicular and parallel respectively to AB, and make 3^14^1 equal to two parts of AD and 4^15^1 equal to one part of AD; then 1^1, 2^1, 3^1, 4^1, and 5^1 are the required centres. For, distances 2^13^1 plus twice 3^14^1, plus $4^15^1 = AD$; then a tangent to ed at e is also a tangent to the

curve *DE*. Actually the point of intersection is a little above *e*, since *De* is rather more than a quadrant, but the point can be adhered to accurately by taking 5^1, the last centre, a little nearer to *d*. In practice, however, the point of intersection is not required to be in any exact position and the method shown is as near accurate as the nature of the problem requires, without being elaborate or involved. It must be remembered that only the outline of the scroll is shown in the figure, and that when the rail is moulded, the members, drawn from the same centres, give a more pronounced scroll effect. This is shown in Fig. 55.

When a more open scroll is required, the distance between a tangent to the first quadrant of the inside curve and the fifth quadrant of the outside curve, marked *x* in Fig. 55, can be governed by adding the desired distance *x* to the given width of the rail and including this in the eight equal divisions. In Fig. 55 *x* was required to be one seventh of *AD*; then the distance *D*8 was set off equal to this amount and *A*8 divided into eight equal parts. The centres were then found as in the previous example; thus distances $C1^1 = A2$, $1^12^1 = A4$, $2^13^1 = A3$, $3^14^1 = A2$, and $4^15^1 = A1$.

If the student has studied Figs. 47, 48, and 49, he should be able to control the scroll curves to suit his own particular tastes and requirements, by varying the number of parts into which the width of the rail is divided.

CHAPTER VI

GRAPHICAL TREATMENT OF CONIC SECTIONS

The three plane curves, the ellipse, the parabola, and the hyperbola, are frequently involved in the practical side of building, and when, as frequently happens, only minimum data are available, a fairly extensive knowledge of the properties of these curves is necessary to obtain the desired result.

Before considering the curves as plane sections of a cone, it is desirable to deal with them as loci of points moving in a plane. A point moving in a plane in such a manner that its distances from a fixed point and a fixed straight line in the plane bear a constant ratio will trace out a conic. When the ratio is unity, the locus of the point is a parabola, when the ratio is less than unity, an ellipse; when greater than unity, a hyperbola. The ratio is called the eccentricity of the conic.

Illustration. *Fig.* 56.—Let F be the fixed point and DD the fixed straight line. Then, because (by construction explained in problem 20) $FA : AO = Fx : FO = Fq : PO = FA^4 : A^4O$, A, x, q, A^4 are points in an ellipse. Similarly, because $FA^1 = A^1O$, $Fy = FO$, and $Fr = PO$, A^1, y, r are points in a parabola. Also, because $FA^2 : A^2O = Fz : FO = Fs : PO$, A^2, z, s are points in a hyperbola.

For each curve the fixed straight line DD is the directrix, and the fixed point F is a focus.

A parabola has one focus and one directrix.

An ellipse and a hyperbola have each two foci and two directrices.

The straight line through the focus, perpendicular to the directrix of a conic, is called the transverse axis, and the figures are symmetrical about this axis.

The ellipse and the hyperbola have each a second axis of symmetry which intersects the transverse axis at right angles in its centre.

Problem 25. *Fig.* 56.—**Given the focus, directrix, and the eccentricity of a conic, to determine points in the curve.**

Let DD be the given directrix, F the given focus, and $\frac{PQ}{PO}$, or $\frac{PR}{PO}$, or $\frac{PS}{PO}$ be the given ratio. For purposes of comparison, curves of all three ratios are drawn in the figure.

Draw a straight line through F, perpendicular to DD and intersecting it in O. Draw PS perpendicular to OF at any convenient distance from O, and on it mark the points Q, R, and S, so as to obtain the given ratios. For the ellipse PQ is less than PO; for the parabola PR is equal to PO; for the hyperbola PS is greater than PO. Draw the straight lines OQ, OR, and OS, and draw FK perpendicular to OF to

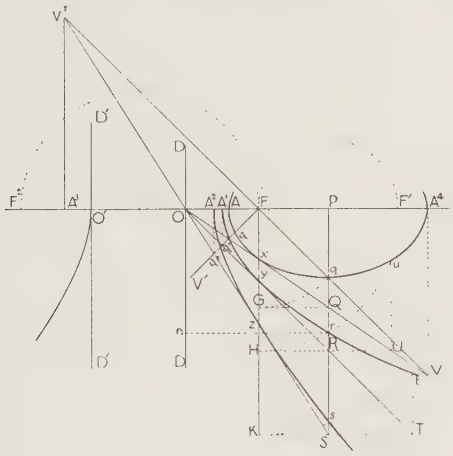


Fig. 56.

cut them in x, y, and z; then x is a point in the ellipse, y a point in the parabola, and z a point in the hyperbola. Draw $F\,V^1$ at 45° to FO, to intersect OQ, OR, and OS in points a, a^1, and a^2; then perpendiculars to OF from a, a^1, and a^2 meet OF in A, A^1, and A^2, the vertices of the conics, and a straight line through F, perpendicular to $F\,V^1$, meeting OQ in V, and OS, produced in V^2, will give the positions of A^4 and A^3, since $A^3\,V^2$ and $A^4\,V$ are parallel to DD. A^3 is the second vertex of the hyperbola, and A^4 is the second vertex of the ellipse. Any number of points in each curve may now be determined as follows: Draw any line perpendicular to DD to cut FK in, say, H, OR in R, and OQ in U. With F as centre, FH as radius, draw an arc to meet perpendiculars to HU from R and U in points r and u; then r is a point in the parabola and u is a point in the ellipse. Other points similarly obtained are s, t, and q.

Consider the point r in the parabola. $PR = PO$, the given ratio, and $FH = PR$, and $rn = PO$. Since r and H are in the arc, centre F, $Fr = FH = PR$, and therefore $Fr : nr = PR$, PO. Any other point in either curve may be shown to be in the position required by the given ratio.

The second directrix of the hyperbola and the other foci F^1 and F^2 are found by making A^3O^1 equal to OA^2, A^3F^2 equal to A^2F, and A^4F^1 equal to AF. The ellipse has a second directrix, not shown, parallel to DD and distant from A^4 an amount equal to AO.

THE ELLIPSE

The straight lines joining the focal points to any point in the curve of an ellipse are called focal distances. The sum of the focal distances is constant and equal to the transverse, or major, axis.

Illustration. *Fig.* 57.—Let AA^1 be the major axis, BB^1 the minor axis, and F, F^1 the focal points of any ellipse; then B^1F and B^1F^1 are each equal to half the major axis, CA or CA^1, therefore $B^1F + B^1F^1 = AA^1$. Now let P be any point in the elliptical curve, then $PF + PF^1 = AA^1$. The string method of constructing an ellipse is based upon this property and should be well known to the advanced student.

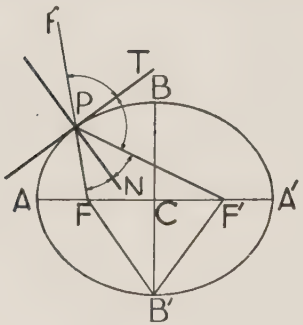


Fig. 57.

The normal and tangent to an ellipse bisect the interior and exterior angles between the focal distances respectively.

Illustration. *Fig.* 57.—Let PN be a normal and PT a tangent to the ellipse at P; then PN bisects the angle FPF^1, and PT bisects the angle fPF^1.

The properties of the ellipse illustrated in Figs. 58 and 59, and explained in the following paragraphs, will be more readily understood if the figures are considered as projections of circles and the lines in the figures are referred to corresponding lines in the circle.

The important rule of projection, *parallel lines project parallel and in their true ratios*, should also be remembered.

The common bisector of any two parallel chords of an ellipse passes through the centre of the figure.

Illustration. *Fig.* 58.—Let op and o^1p^1 be any two parallel chords of an ellipse, and let st be the common bisector of these chords; then the middle point of st is in C, the centre of the ellipse.

The principal axes of an ellipse bisect the angle between any two equal diameters.

Illustration. *Fig.* 58.—Let st and uv be any two diameters of equal length; then AA^1 bisects the angles vCt, sCu, and BB^1 bisects the supplementary angles vCs, tCu.

An ellipse may be inscribed in any parallelogram and any number of points in the curve may be determined by intersecting lines.

Illustration. *Fig.* 59.—Let $QRST$ be any parallelogram, and a, a^1, b, b^1 be the middle points of the four sides; then, if the distances $a1$, 12, $2c$ are equal, and $a1^1$, 1^12^1, 2^1Q are also equal, the straight lines b^11, $b1^1$, and b^12, $b2^1$ will intersect in points on the curve, and the ellipse will be tangent to the sides of the parallelogram at a, a^1, b, b^1. Other points in the curve may be found in a similar manner, as shown at p and q. Points m and n, Fig. 58, are determined in the same way.

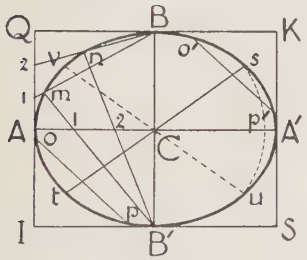


Fig. 58.

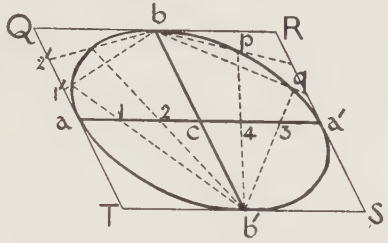


Fig. 59.

If any circle containing any two mutually perpendicular diameters be projected into an ellipse, the projections of the diameters of the circle are called conjugate diameters of the ellipse, and each conjugate diameter is the bisector of all the chords parallel to the other conjugate diameter.

Illustration. *Fig.* 59.—Consider the figure as the projection of a circle inscribed in a square; then aa^1 and bb^1 are the projections of a pair of mutually perpendicular diameters of the circle, and are therefore conjugate diameters of the ellipse; all chords parallel to aa^1 will be bisected by bb^1, and all chords parallel to bb^1 will be bisected by aa^1.

The diagonals of any rectangle circumscribed about an ellipse are conjugate diameters.

For any pair of conjugate diameters aa^1, bb^1, tangents to the ellipse at a and a^1 are parallel to bb^1 and tangents at b and b^1 are parallel to aa^1.

Problem 26. *Fig.* 60.—**Given any pair of conjugate diameters of an ellipse, to determine the principal axes.**

Let *aa* and *bb* be the given diameters and *C* their point of intersection.

Draw *be* perpendicular to *aa* and equal to *Ca*. Join *eC* and draw a circle with *Ce* as a diameter, centre *o*. Through *b* and *o* draw a straight line to cut the circle in *n* and *m*; then the directions of the required axes are, major axis through *Cn*, and minor axis through *Cm*; *CA*, the semi-major axis, is equal to *bm*, and the semi-minor axis *CB* is equal to *bn*. If it is required to draw the curve, the tangents *da* and *db*, parallel to *bb* and *aa*, will be found useful as directors of the curve.

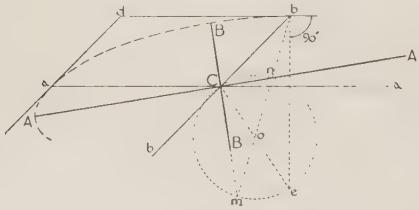


Fig. 60.

Theorem.—The major circle of any ellipse bisects all the straight lines drawn from a focus to the circumference of a circle whose centre is in the other focus and whose radius is equal to the major axis.

Illustration. *Fig.* 61.—Let *F*, F^1 be the foci of the ellipse AA^1, BB^1, and let *D* be in the circumference of a circle whose centre is in *F* and whose radius is equal to AA^1; then F^1D is bisected by the major circle in *s*. Similarly, *FE* is bisected by the major circle in *u*, when *E* is on the circumference of a circle, centre F^1 and radius AA^1.

Let the circles *Dm* and *Ep*, Fig. 61, be called the focal circles of the ellipse; then if any two radii *FD*, F^1E of the focal circles be drawn to intersect in any point *P* on the curve, the tangent to the ellipse at *P* bisects the distances *EF*, DF^1 perpendicularly, the points of bisection *s*, *u* being in the circumference of the major circle, and the points *m*, *n*, *o*, *p*, in which the tangent cuts the focal circles, are such that $Em = En = EF^1 = Do = Dp = DF = AA^1$, and the normal at *P* is parallel to *EF* and DF^1. Since the sum of the focal distances is constant and equal to the major axis, then $PE = PF$ and $PD = PF^1$ and the angles *EPu*, sPF^1 are equal, and the angles *sPD*, *uPF* are equal; therefore the angles *DPs*, F^1Ps, *EPu*, *FPu* are equal, and the tangent bisects the exterior angles between the focal distances. Also, since DF^1 is bisected perpendicularly by the tangent, the

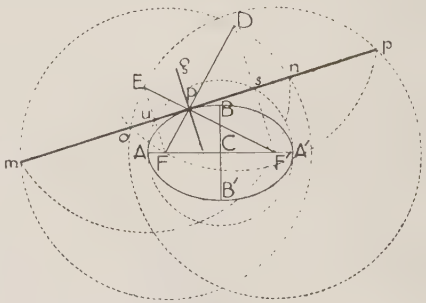


Fig. 61.

circular arc, centre D, radius DF, will cut the focal circle, centre F^1, and of equal radius, in points o and p on the tangent. Points m and n on the arc, centre E, radius EF^1, may be shown in the same way to be in the tangent.

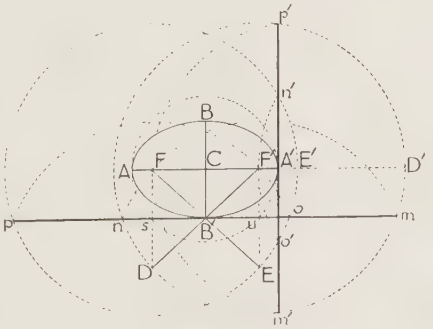


Fig. 62.

Fig. 62 illustrates the same properties of the major and focal circles by tangents at the extremities of the axes. When the tangent is at B or B^1, FD and F^1E are perpendicular to AA^1; when the tangent is at A or A^1, E^1, L^1, corresponding to E and D in the figure, are on the major axis produced. D is the centre of the arc pF^1o, E is the centre of nFm. D^1 is the centre of $o^1F^1n^1$, E^1 is the centre of p^1Fm^1.

In Fig. 63 some further properties of the ellipse are illustrated. First consider the ellipse A, A^1, B, B^1 as a projection of the major circle. If the circle is turned about the diameter AA^1 so that BB^1 becomes the projection of bb^1, any point s on the circle will project into a point T

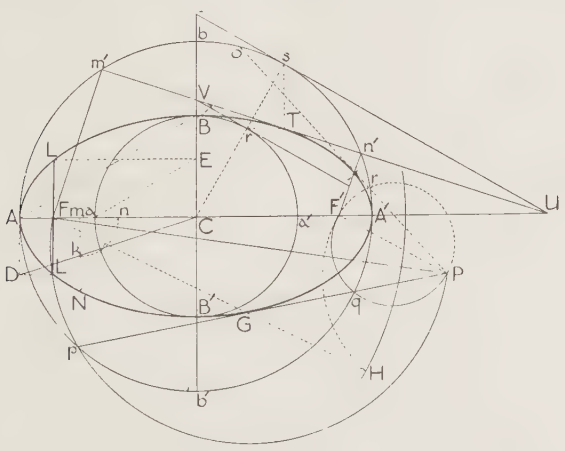


Fig. 63.*

on the ellipse, and sT will be perpendicular to AA^1; a tangent to the circle at s will project into a tangent to the ellipse at T, and the two tangents will intersect on AA^1, produced, in U, since U is in the fixed line about which the circle was turned. Similarly, if the minor circle

be turned about the diameter BB^1, so that AA^1 becomes the projection of aa^1, any point r will project into a point T on the ellipse, and the tangent Vr will project into the tangent VT, V being a point in the fixed line BB^1, produced; and Vr, sU will be parallel, since they are both perpendicular to Crs.

The feet of perpendiculars to any tangent from the foci are in the major circle; therefore if Fm^1 and F^1n^1 are perpendicular to the tangent VT, m^1 and n^1 are in the major circle.

Problem 27. *Fig.* 63.—**To draw tangents to an ellipse from any point outside the curve.**

Let P be the given point, F and F^1 the foci of the given ellipse. Join P, F and P, F^1, and on these lines as diameters draw circles to cut the major circle in points o, p, q, r; then Pqp, Pro are the two tangents required. To determine the point of contact, take P as centre, PF^1 as radius, and draw an arc to intersect the focal circle, centre F, radius AA^1, in H; draw FH to cut the ellipse in G; then G is the point of contact. Note that GH equals GF^1, F^1q equals Hq, and F^1H is bisected perpendicularly by PG in q. Compare with Fig. 61, and verify G by the second focal circle and focus.

The latus rectum of a conic is the chord through the focus perpendicular to the transverse axis, and the latus rectum of an ellipse is a third proportional to the major and minor axes.

Illustration. *Fig.* 63.—LL is the latus rectum and is perpendicular to AA^1 through F; if aE be drawn parallel to AB, EC equals LF, and $LF : CB = BC : AC$; therefore $LL : BB^1 = BB^1 : AA^1$.

If from any point m in the major axis of an ellipse a perpendicular be drawn to cut the curve in a point N, the normal at N will meet the major axis in a point n, such that mn is to mC as FL is to AC, where AC is the semi-major axis and FL the semi-latus rectum.

Illustration. *Fig.* 63.—AD is parallel and equal to FL; mN is perpendicular to AC and cuts DC in k; nN is a normal to the curve at N; them $mn = mk$ and $mn : mC = FL : AC$. This property is very useful in cases where a large number of normals are required, as in an elliptical arch.

ELLIPTICAL TRAMMELS

One of the best practical methods of drawing an ellipse is by means of a trammel, and is based on the following theorem.

Theorem.—If any straight line containing three fixed points be moved so as to have two of the points each always in one of two fixed straight lines, then the locus of the third point will be an ellipse or a circle. The

locus will be a circle only when the locus point is equidistant from the two other points and the two fixed lines are perpendicular to each other.

Illustration. *Fig.* 64. Let AA^1, BB^1 be the principal axes of any ellipse, then the straight line $R^1P^1Q^1$, where R^1P^1 equals BC, and P^1Q^1 equals AC, will have P always in the curve of the ellipse when R moves always in AA^1 and Q moves always in BB^1; the lines produced where necessary. One position of the line is shown at RPQ. This line is called the long trammel to distinguish it from the short trammel shown at $r^1p^1q^1$ in the same figure. In the short trammel, r^1p^1 equals AC, while r^1q^1 equals AC minus BC. In using the short trammel, r must be always in BB^1, q always in AA^1; then p is always in the curve. One position of the short trammel is shown at rqp.

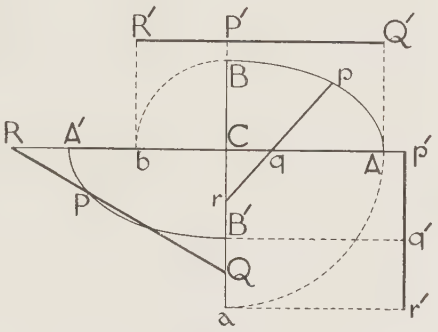


Fig. 64.

For constructing a small ellipse by the trammel method, the straight line containing the three points should be drawn on tracing paper or tracing cloth, and a series of positions of the locus point pricked through on to the drawing paper. For large ellipses the methods illustrated in Fig. 65 are more suitable. The long trammel is more accurate in working than is the short trammel, and should always be used in cases where there is little difference between the lengths of the axes.

Problem 28. *Fig.* 65.—**Given one of the axes, and a point in the curve of an ellipse, to determine the other axis and two trammels for drawing the curve.**

Let either AA^1 or BB^1 be the given axis, and P be the given point.

Draw the perpendicular bisector of the given axis, and with centre P, and radius equal to half the given axis, cut the other axis in points rQ or Rq. Then R, P, Q will be in the same straight line, and P, q, r will also be collinear. These lines being the required trammels, each in the position required for tracing the curve through P, the long trammel will have R always in AA^1, and Q always in BB^1, produced beyond B and B^1 a distance equal to CA, and beyond A and A^1 a distance equal to BC. To determine the length of the second axis without drawing the curve, make CA and CA^1 each equal to PQ or Pr, and CB, CB^1 each equal to PR or Pq.

In the theorem upon which the trammel is based it was stated that when the locus point is equidistant from the two other points in the moving line, and the fixed lines are at right angles, the locus is a circle ; in the figure it is shown how this may be made use of to modify and improve either the long or short trammel. First consider the long trammel QPR. If s is the mid-point of QR, then, while P is tracing the ellipse, s will be moving always in the plane circle ss^1, centre C ; therefore, by using a radius bar to connect C and s, one end of the trammel may be dispensed with ; for, s being constrained by the radius bar to move always in the circle ss^1, one of the points Q or R will be sufficient to keep P moving always in the ellipse, since $Qs = Rs = Cs$, and Qs, Rs are always in the same straight line. If, in conjunction with the radius bar, a straight edge be used to guide the second point Q or R, the curve will be determined more quickly and accurately than is possible by any other method. For large ellipses, or where several similar curves are required, the method just described is particularly suitable. In the short trammel Pqr the point u is equidistant from q and r, and will therefore have for its locus the circle centre C, radius Cu ; then C, u may be connected by a radius bar, and the guiding point r may be dispensed with.

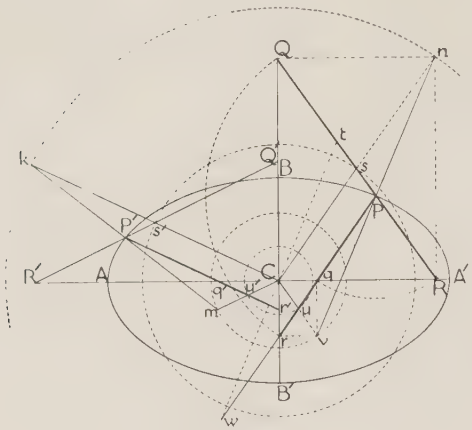


Fig. 65.

Normals determined from Trammel Positions.

Referring again to Fig. 65, let QPR be any position of the long trammel, and Qn, Rn be perpendicular to the axes, then nP is a normal to the curve at P. Also if from r and q in the short trammel, rv, qv be drawn perpendicular to the axes, then vP is a normal at P and vPn is a straight line. The diagonals of any parallelogram bisect each other, therefore, s being the mid-point of QR, Cs produced will always pass through n, and sn will be equal to Cs ; then by making the radius bar extend to n, the trammel may be stopped at any position and the

normal drawn through nP. In the same way the radius bar Cu may be extended to v, uv being equal to uC, and the normal vP may be drawn for every position of the short trammel.

Alternative methods of determining normals from the trammel without the use of a radius bar are as follows: Make st equal to sP, then the normal at any point P will be parallel to a line through tC. For the short trammel, make rw equal to Pq, then Cw is always parallel to the normal at the point P through which the trammel is moving.

With centre C draw circles through v and n, then for every position of the long trammel a normal may be drawn thus: Let $R^1P^1Q^1$ be one position of the trammel. Draw Cm parallel to R^1Q^1, then the straight line through mP^1 is a normal at P^1. Again, let $P^1q^1r^1$ be one position of the short trammel. From C draw a straight line parallel to P^1r^1 to cut the circle through n in k, then kP^1 is a normal to the curve at P^1.

The student should prove the following equalities for every position of the two trammels, as, by so doing, other useful possibilities of the practical trammel may be seen and the risk of forgetting this useful construction will be decreased. $Pw = RQ$; $Rt = Pr$; $Cv = Pt$; $Pv = Ct$; $Cw = Ct$; $Pu = Qs$; $Pu = Cs$; and $qv : rv = Rn : Qn$.

When the lengths of the two axes of an ellipse are known, any number of trammels may be found to trace the curve.

Illustration. *Fig.* 66.—Let C be the centre, CA and CP the semi-axes of any ellipse, and let PD equal CA. Then if a circle be drawn with CD as diameter, any chord of that circle passing through P will be a trammel for the ellipse, and the straight lines through C, R and C, Q will be the guiding lines, R and Q being the extremities of the chord. Two positions of the trammel are shown at rpq and $r^1p^1q^1$.

Since any number of chords may be drawn through P, there are any number of trammels, but each trammel will have its own guiding lines. When the chord is at right angles to CD, as in Fig. 67, the trammel is equi-ended, that is $RP = QP$. This trammel is of special value because it cannot be wrongly applied; this is important when several people are required to use the same trammel.

The principle upon which the trammels in Figs. 66 and 67 are based is as follows:

If a circle of radius r roll inside a fixed circle of radius R, where $r = \frac{R}{2}$, then the locus of any point in the circumference of the rolling circle is a diameter of the fixed circle, and the locus of any other point, excepting the centre, in the plane of the rolling circle is an ellipse. The locus of the centre is, of course, a circle. Note.—When the

diameter of the rolling circle is greater or less than half the diameter of the fixed circle, the locus of any point in the circumference is a hypocycloid, and the loci of other points in the plane of the circle are hypotrochoids. Strictly speaking, the straight line and the elliptical loci are special cases of these curves. In Fig. 66 the radius cD of the smaller circle is half CD, the radius of the larger circle ; therefore, if the smaller circle is made to roll completely around the larger circle the loci of the points Q and R will be the straight lines FF^1, EE^1, and the locus of P will be the ellipse ; then by moving the straight line RQP, with Q always in FF^1 and R always in EE^1, the movement of P in the line will be identical with the movement of P in the rolling circle, and RQ may always be considered as a chord of that circle.

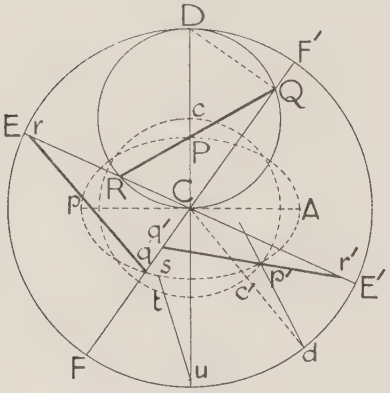


Fig. 66.

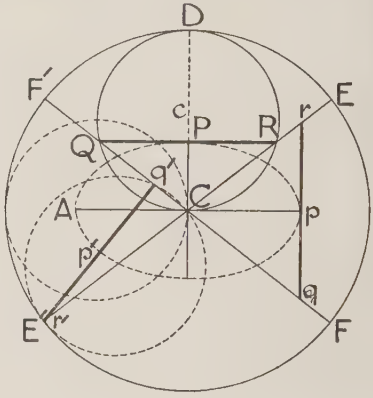


Fig. 67.

Since the point of contact of the two circles is always the instantaneous centre of rotation of the point P, the normal for any position of the trammel will be a straight line through P and the corresponding point of contact between the circles. For example, let $q^1p^1r^1$ be any position of the trammel, and ctc^1 the circular locus of c, the centre of the rolling circle ; then, if p^1c^1 is equal to Pc, the straight line through C, C^1, the centres of the two circles, will cut the fixed circle in d, the point of contact of the circles corresponding to the position of $q^1p^1R^1$, and the line dp^1 will be a normal to the ellipse at p^1.

The line DcP may also be used to trammel the given ellipse, c being always in the circle ct and D always in the straight line through DC. In using this trammel, the point c should be controlled by a radius bar Cc. The line uts shows one position of this trammel.

There are any number of triangular trammels that will give any required ellipse, but the principle is the same as that of the straight

line trammel. Consider the triangle DPQ; since D is a point in the circumference of the rolling circle, the straight line through DC will be its locus, then, by keeping D always in DC and Q always in FF^1, P will be in the ellipse as before. If the point c were marked on the trammel, the trammel could be arrested at any point and a normal determined at the position of P, by laying a straight edge to pass through Cc, and marking the point d in the fixed circle from which the normal may be drawn.

Centres and Radii of Curvature.

The centre of curvature at any point in the circumference of an ellipse is the centre of the circular arc whose curvature most nearly conforms to the curvature of the ellipse at the point. Mathematically considered, no appreciable length of an elliptical arc is identical with any circular arc; nevertheless, if sufficient centres and radii of curvature for points in the ellipse are determined, the variation between the true ellipse and a curve made up of tangential circular arcs will lie within the breadth of a reasonably fine line.

Illustration.—In Fig. 68 the points r, s, t, u are the centres of curvature of the ellipse at A, Q, P, and B respectively, and the four centres are transferred to Fig. 69. The manner of determining the centres will be explained later. In the two figures corresponding distances are indicated by corresponding letters. In Fig. 69 circular arcs are drawn from the centres u, t, s, r through B, P, Q, and A, the minimum distances between adjacent arcs being at the positions marked d, e, f. Although the diagram is necessarily to a small scale, it should convey some idea of the nature and magnitude of the discrepancies between the circular arc and the true ellipse.

Problem 29. *Fig.* 68.—**To determine the centre and radius of curvature for any point in an ellipse.**

Firstly, when the point is in the extremity of an axis. Let A or B be the given point. Draw Ab and Bb perpendicular to AC and BC. Join AB. From b perpendicular to AB draw a straight line to cut AA^1 in r, and BB^1, produced, in u; then r is the centre and rA the radius of curvature at A, and u is the centre, uB the radius of curvature at B.

Secondly, when the point is not in the extremity of an axis.

Let P be the given point, and f the focus of the ellipse.

Draw a normal to the curve at P to cut AA^1 in a point m. From m, perpendicular to Pm, draw a straight line to cut a straight line through Pf in n. Draw nt perpendicular to Pn intersecting the normal

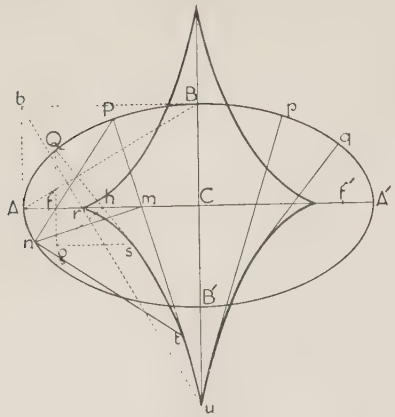


FIG. 68.*

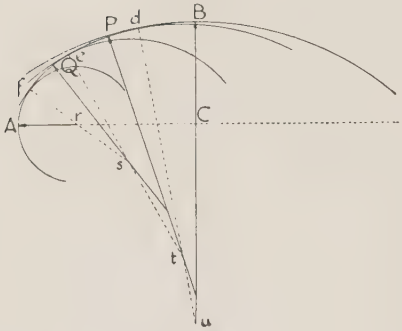


FIG. 69.*

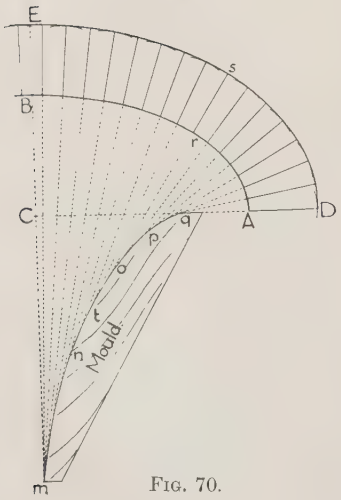


FIG. 70.

Pm, produced, in *t*; then *t* is the centre, and *tP* the radius of curvature at *P*. *s*, the centre of curvature at *Q*, is determined in the same way.

The locus of the centres of curvature of an ellipse is called the evolute, and is a curve such that all normals to the ellipse are tangents to the evolute. The complete evolute is shown in Fig. 68, the four sides being equal in every respect. The evolute is symmetrical about the axes of the ellipse.

In Fig. 70 the evolute is made use of to draw the joint lines in the elevation of an elliptical brick arch. The edge of the mould is shaped to the curve of the evolute, and after dividing the outer curve into a number of equal parts to suit the size of the voussoirs, a straight edge is moved into the positions shown by the dotted lines, and the joint lines of the arch drawn. In practice the mould would be of wood and lightly nailed in the required position on the setting out board. The five centres of curvature *m*, *n*, *o*, *p*, *q* shown in the figure will be found sufficient for drawing the curve reasonably accurate, if a flexible rod is bent to pass through the points. A piece of thin lead pipe can be made to serve for the evolute mould, if sufficient care is taken in binding the pipe and applying the straight edge.

It is necessary at this point to mention that *no two conics are parallel*. In Fig. 70 the curve AB is the true ellipse and the parallel DE is determined by drawing a number of circular arcs with centres in the curve AB and radius equal to the width of the arch face, and drawing DE to touch the arcs, as shown.

The Ellipse as a Section of a Cylinder.

Fig. 71 is of particular interest to students of geometrical handrailing and is intended to explain some of the operations involved in the determination of face moulds. Let the figure $ADFC$ be the plan, and OO^1FF^1 the elevation of one quarter of a hollow cylinder. Suppose the quarter cylinder to be cut by the plane *vth*; then the true shape of the section is bounded by the elliptical quadrants A^1D^1, C^1F^1 and the straight lines A^1C^1, D^1H^1, and level ordinates across the section plane are parallel to O^1C^1, since OC is parallel to the horizontal trace of the section plane. Therefore O^1A^1, A^1B^1, B^1C^1 are equal to OA, AB, BC respectively, and D^1H^1 equals DH, and E^1c^1 equals Ec. EB is the plan of the centre layer of the cylinder, $E^1b^1B^1$ being the true shape of its section. It has already been stated that parallel lines project parallel; it follows that the three systems of parallel chords of the plan circles will project into three corresponding systems of parallel chords of the ellipses in the developed section. It will be seen, then, that having found E^1, B^1, b^1, A^1, and C^1, the points D^1, F^1, a^1, and c^1 may be determined by drawing parallel lines.

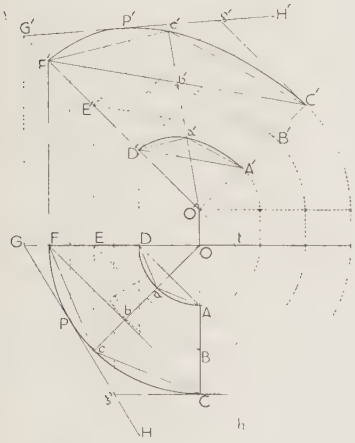


Fig. 71.

In the square cut or tangent system of handrailing, the centre layer of the cylinder containing the wreath is dealt with, so as to give the necessary data for the elliptical outlines of the face mould with the minimum of work. This will be discussed more fully in a later chapter.

The development of tangents to the plan curves is also an important part of geometrical handrailing, particularly when the wreath is being worked and set out by what is known as the method of normal sections. Consider the tangents to the circular arc FC, the points of contact being P and C. Then because GP is tangent to the circle at P, G^1P^1

will be a tangent to the ellipse F^1C^1 and the point of contact P^1 is the development of P. Also the tangent gC develops into the tangent g^1C^1, the point of intersection g of the two tangents in plan, having g^1 for its development.

THE PARABOLA

In the early part of this chapter the term eccentricity, as applied to a conic, is explained and a parabola is defined as the locus of a point moving in a plane in such a manner that its distances from a fixed point and a fixed straight line in the plane are always equal; that is, the eccentricity is $\frac{1}{1}$ or unity. Arising out of this basic property of equality, many other important distances connected with the curve are similarly related, and consequently the parabola is more easily dealt with graphically than are the ellipse and the hyperbola. The simplest and most accurate method of constructing the curve when the directrix and focus are given is shown to the left of the axis in Fig. 72. AA is the axis, DD the directrix, and F the focus. $O5$ is any straight line perpendicular to DD and $F5^1$ is any straight line drawn from F. $O1$ equals $F1^1$, $O2$ equals $F2^1$, and so on to $O5$, $F5^1$. Point E is the intersection of DD and AA, and V, the vertex of the parabola, is the mid-point of FE. In the figure, $O1$, $F1^1$ are made equal to EV and FV, therefore the circular arc, centre F, radius $F1^1$, will meet the parallel to DD through point 1 in V. Other points in the curve are found in a similar manner, thus: parallels to DD from points 2, 3, 4, 5 intersect circular arcs, centre F, through 2^1, 3^1, 4^1, 5^1 in points a, b, c, and d on the parabola.

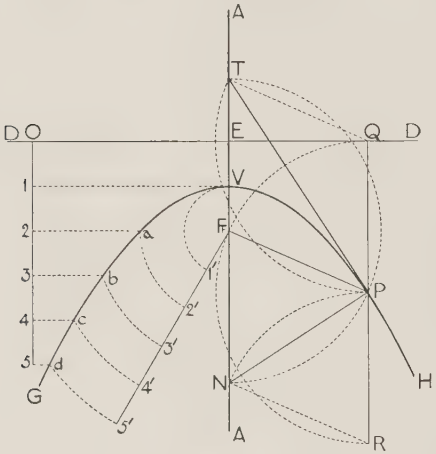


Fig. 72.

Tangent and Normal to a Parabola.—A parabola is symmetrical about its axis, and in Fig. 72 the two branches VG, VH are symmetrical about AA. Let PT be a tangent and PN a normal at any point P on the curve; then TPN is a right angle. Let T and N be the points where the tangent and normal meet the axis, and let QP parallel to AA be produced to R, so that $PR = PQ$. Join QT, PF, RN; then $TF = NF = QP = RP = TQ = FP = NR$, and QF equals

PN; therefore the two rhombic figures $PQTF$, $PRNF$ are congruent, the tangent PT being the long diagonal of one figure, the normal PN the short diagonal of the other figure; then the tangent bisects the angle QPF and the normal bisects the angle FPR.

The parabola is the limiting position between the elliptic and hyperbolic sections of a cone, and may for certain purposes be considered as an extreme case of either. Consider Fig. 73 in which BB^1 represents a double cone, vertex A, cut by three section planes, EE^1, PP^1, and HH^1, all passing through the point V. Because the plane EE^1 cuts all the generators on one side of the vertex of the cone, the section is an ellipse; because the plane PP^1 is parallel to one generator of the cone, the section is a parabola; and because the plane HH^1 cuts both branches of the cone, the section is a hyperbola.

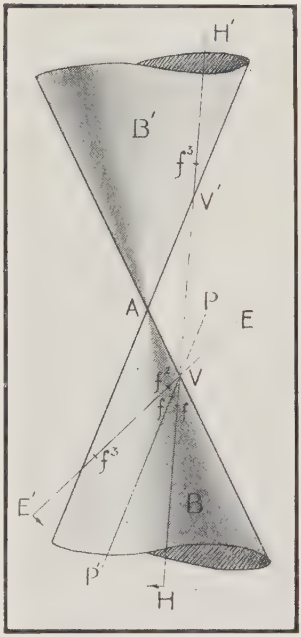


Fig. 73.

Now suppose EE^1 to turn about V in the direction indicated by the arrow; then the section will continue to be elliptic until the instant when it coincides with PP^1, at which instant it is parabolic. In the same way if HH^1 turn about V as indicated, the section will continue to be hyperbolic until the instant that it coincides with PP^1, when it is again parabolic. V is the common vertex of all the sections indicated in the figure and throughout the turning movements of the two planes; as long as the section is elliptic there are two foci on the same side of V, but when the section is hyperbolic, the two foci are on opposite sides of V. Therefore, if the parabola is regarded as an extreme case of the ellipse, a second focus may be imagined at infinity on the same side of the vertex as is the actual focus. Conics, as sections of a right circular cone, are more fully dealt with at the end of this chapter.

Refer again to Fig. 72, in which the normal PN bisects the angle FPR. Since PR is parallel to AA the two lines may be considered as meeting at infinity in the direction from P to R; then PF is one focal distance of P and PR the direction of the second focal distance from the imaginary focus, and the normal, PN, bisects the interior angle

between the focal distances as in the ellipse. If the parabola (same figure) now be considered as an extreme case of the hyperbola, the second focus may be imagined as being in AA produced to infinity in the direction of F to V, and FPQ becomes the angle between the focal distances. In the hyperbola the *tangent* bisects the interior angle between the focal distances, and it will be seen that PT may be considered as fulfilling the same condition.

Some further important relationships between normals and tangents to a parabola are shown in Fig. 74. AA is the axis, F the focus, D the directrix, V the vertex, and LL the latus rectum. Let P be any point on the curve, PT a tangent and PM a normal to the curve at P; let PS be a perpendicular to AA from P. Then $VT = VS$, and $SM = FD = FL$; for $FT = FM$ and $VF = \frac{1}{2}FD$, therefore $TM - TS = 2\,VF$, and because L is in the curve and F is the focus, and the directrix D is parallel to LL, $FL = FD$. SM is called the subnormal, and, from the foregoing, *the subnormal is constant and equal to the distance between the focus and the directrix.*

A tangent to the curve at V intersects any tangent TP in a point R, such that a perpendicular to the tangent at R will pass through the focus, and since FR is parallel to PM and $TF = MF$, $TR = PR$ and $PM = 2FR$. Let QP be parallel to AA, Q being in the directrix, then because $PF = PQ$ and PT bisects the angle FPQ, FR produced will pass through Q.

Any pair of mutually perpendicular tangents intersect on the directrix, and the straight line joining their points of contact is a focal chord. Thus in Fig. 74 NU is perpendicular to the tangent PT, from the point N in the directrix, therefore NU is tangent to the curve at U and the straight line joining P to U passes through the focus.

If the points of contact and the point of intersection of any two tangents be joined to the focus, the angles subtended by the tangents at the focus are equal to each other and to the exterior angle between the tangents. Thus, in Fig. 74 the angles NFU, NFP, TNU are equal; in this particular case each angle is a right angle.

The points of intersection of any three tangents to a parabola lie in the circumference of a circle through the focus. Thus, in Fig. 75 the points P, M, N are the intersections of the tangents to the curve at Q, R, and V; then P, M, N, F are concyclic.

The feet of perpendiculars to the directrix from the points of contact of any two tangents to a parabola are in the circumference of the circle which passes through the focus and has its centre in the point of intersection of the tangents. Thus, in Fig. 75, r and q, the feet of perpendiculars

to the directrix DD from R and Q, are in the circumference of the circle, centre P, radius PF.

Problem 30. *Fig.* 76. **—Given the double ordinate AB, and its abscissa EV, of a parabola, to determine points in the curve by intersecting lines, and to determine the focus, directrix, and a normal and tangent at any point P.**

Draw CD through V parallel to AB, and draw AC, BD perpendicular to AB. Divide AC and AE into any number of equal parts, as C-1, 1-2, 2-3, 3-4, 4-A and E-1^1, 1^1-2^1, 2^1-3^1, 3^1-4^1, 4^1-A. From 1^1, 2^1, 3^1, 4^1 erect perpendiculars to intersect radials from V to 1, 2, 3, 4; then the points of intersection are points in the parabola which will pass through A and be tangent to CD at V. Points on the other side of the axis are shown determined in the same way. From any point P draw PR parallel to AB intersecting EV in R. Produce EV to T, making VT equal to VR; draw TP. Then TP is a tangent to the curve at P. Draw a perpendicular to TP from P intersecting VE, produced, if necessary, in Q; then PQ is a

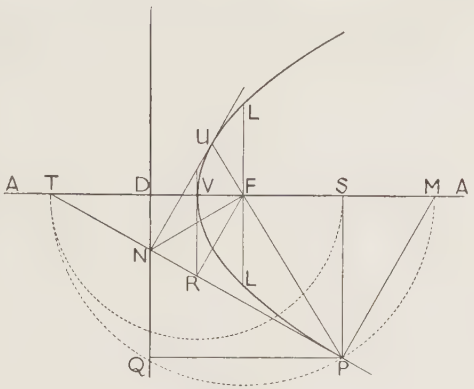


Fig. 74.

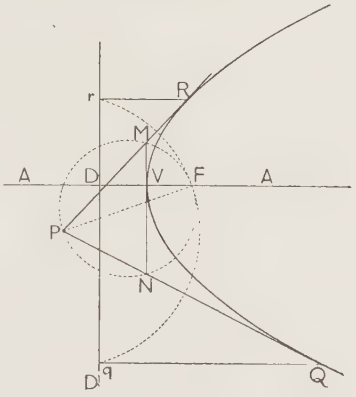


Fig. 75.

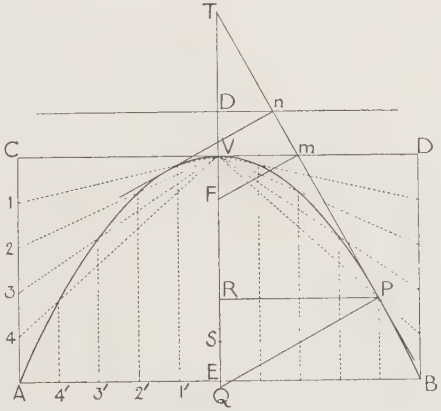


Fig. 76.

normal at P and RQ is the sub-normal. Bisect QR in S and make VF and VD each equal to RS; then F is the focus and a perpendicular to EV through D is the directrix. Or, having determined the tangent to intersect CD in m, draw mF perpendicular to it to obtain F, and n, a point in the directrix, may be found by drawing a second tangent perpendicular to the first, as shown.

Problem 31. *Fig.* 77.—**To determine the evolute of a parabola.**

Let F be the focus, V the vertex, and D the directrix of any parabola, and let p, q, r be any points on the curve. The evolute of a curve has been defined in this chapter as the locus of the centres of curvature of all points in the curve; therefore the problem is that of finding a sufficient number of centres of curvature to allow the evolute to be drawn with reasonable accuracy. The centres of curvature for any conic may be found by the method explained in problem 29, Fig. 68, but the construction is somewhat simplified in the parabola. Consider the point q in the figure. With F as centre, Fq as radius, cut the axis in q^1, then the straight line through q, q^1 is the normal at q and will contain the centre of curvature for that point. Draw the straight line qu through F and make Fu equal to Fq. From u draw a perpendicular to qu to meet the normal in d; then d is the centre of curvature of the curve at q. The points b, c, e are the centres for p, L, and r respectively, and are found in the same way. A second method of obtaining the centres is shown for the point L, and is as follows: Produce the normal to intersect the directrix in M and make Lc equal to $2LM$; then c is the required centre. In each conic the former construction fails when applied at the extremity of an axis, but in the parabola the second method is applicable; thus, make Va equal to $2VD$, that is, $Fa = FV$. A smooth curve through a, b, c, d, e is the evolute of the curve from V to r. The evolute is tangent to the axis at a and is symmetrical about the axis. If Vg be made equal to $4DF$, a perpendicular to the axis from g will pass through the point of intersection of the parabola and its evolute. A series of circular arcs having their centres in the centres of curvature and having the appropriate radii will produce a curve closely resembling the parabola.

Problem 32. *Fig.* 78.—**Given the points of contact and the directions of two tangents to a parabola, to draw the curve and determine the axis.**

Let AB and AC be the given tangents, P and Q the points of contact.

First Method.—Divide AP and AQ into a corresponding number of equal parts, and letter them A-a, a-b . . . eP and Q-a^1, a^1b^1 . . .

e^1A as shown in the figure. Join a-a^1, b-b^1, c-c^1, d-d^1, and e-e^1; then a smooth curve to touch each line, as shown, is the required parabola.

Second Method.—Join P, Q and find the mid-point 3^1. Join 3^1, A and draw $P3$ parallel and equal to half $A3^1$. Divide $P3$ and $P3^1$ into a corresponding number of equal parts, as 1-2, 1^1-2^1. From u, the mid-point of $A3^1$, draw $u1$, $u2$ to intersect parallels to $P3$ from 1^1, 2^1 in s and t; then u, t, s, P are points in the required curve; the construction corresponds to that in Fig. 76.

Straight lines parallel to the axis of the parabola are diameters.

The common bisector of any system of parallel chords of a parabola is a diameter, and its point of intersection with the curve is the point of contact of a tangent drawn parallel to the chords.

Illustration. Fig. 78. —Let tv and sw be parallel chords having p and o for their mid-points; then the straight line through p and o is a diameter, and all chords parallel to tv and sw will be bisected by it. Therefore, since

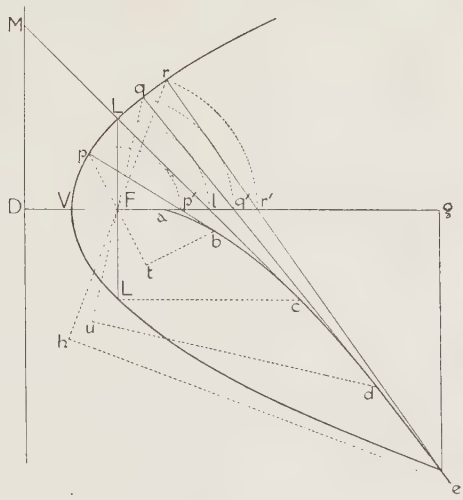


Fig. 77.

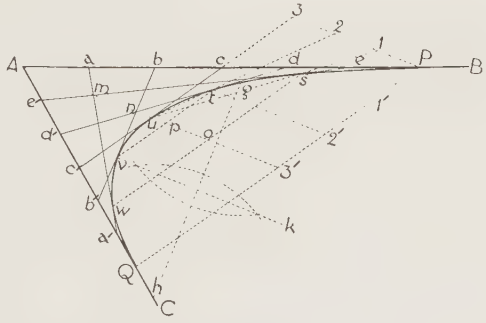


Fig. 78.*

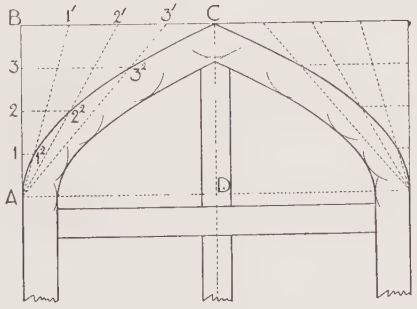


Fig. 79.*

PQ is parallel to *sw*, the diameter through *p* and *o* bisects it in 3^1. Also, if the tangent cc^1 is parallel to *tv* and *sw*, it touches the curve at *u*, the point in which the diameter through *po* meets the curve.

Since diameters are parallel to the axis, the latter may be determined by drawing any chord, as *gh*, perpendicular to any diameter, determined as above, and drawing its perpendicular bisector *vk*; then *vk* is the axis and *v* is the vertex.

Tangents at the extremities of a chord intersect on the diameter, produced, which bisects the chord, and the distance between the point of intersection and the mid-point of the chord is bisected by the curve.

Illustration. *Fig.* 78.—Let *AP* and *AQ* be tangents to the parabola at *P* and *Q*, then *A* is on the diameter $u3^1$ produced, 3^1 being the mid-point of the chord *PQ*, and $A3^1$ is bisected in *u*. Similarly $nu = pu$ and $mu = ou$, since *n* is the point of intersection of tangents at *t* and *v*, and *m* the intersection of tangents at *S* and *W*, *p* and *o* being the mid-points of the chords *tv* and *sw*. This property will give a third solution to problem 32, and is employed in setting out arches and easings in handrails, as shown in later problems.

In Fig. 79 a door-head of pointed outline, made up of two parabolic arcs and their parallels, is shown. The diagram contains all the necessary construction lines, the method of obtaining the outer curves being the same as that explained in problem 30. The parallel curves are obtained by drawing circular arcs, having their centres in the true curves and radii equal to the required distance between the curves.

Problem 33. *Fig.* 80.—**To draw the mould for an easing to connect two straight handrails, the centre line of the easing to be parabolic.**

Let *A* and *B* be the edge views of the ends of the two straight rails which the easing is required to connect. Produce the straight centre lines through *a* and *b* to intersect in *c*. Draw the straight line *ab* and bisect it in *d*. Join *cd* and determine the mid-point 2. Through 2 draw *pr* parallel to *ab*, and draw *a*2, *b*2. Draw *pq* and *rs* parallel to *cd* and find their mid-points 1 and 3. A curve through *a*, 1, 2, 3, and *b* is the required parabolic centre line. The curve must be drawn tangent to *ac* at *a*, *bc* at *b*, *pr* at 2, and lines through 1 and 2 parallel to *a*2 and *b*2. When a curve is to be drawn through a number of points, one or two tangents to the required curve should be determined whenever convenient, as they will give greater assistance in directing a curve than is obtained from double the number of points.

In Fig. 81 the easing is shown with an elliptic centre line, determined as follows: Produce the straight centre lines to intersect in *C*. Draw *bo* parallel and equal to *aC*. Divide *bo* and *bC* into the same

number of equal parts. Draw *aoO*, making *oO* equal to *ao*. Then *a* and *O* are the points from which radiating lines are drawn to intersect in points in the elliptic arc *a*, 1, 2, 3, 4, *b*, as previously explained. In both easings the outline curves are envelopes of a circle the locus of whose centre is the centre line, having a diameter equal to the thickness of the rail.

Problem 34. *Fig.* 82.—**Given the span and rise of a pointed arch, the soffit curves to be equal arcs of parabolas, to determine the curves, parallels, and joint lines.**

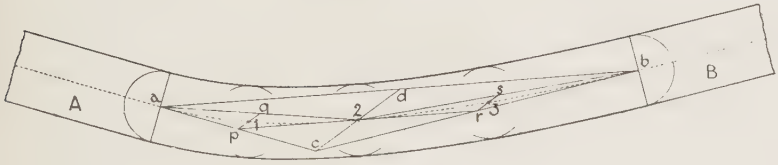


Fig. 80.

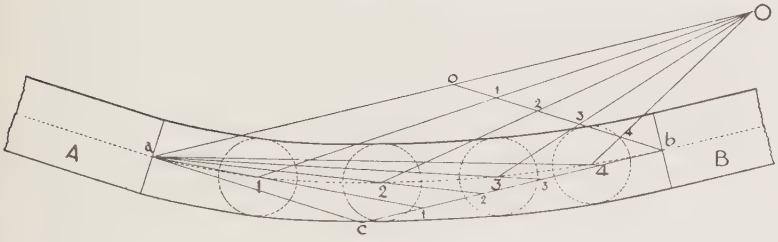


Fig. 81.

Let *AB* represent the given span and *HC* the given rise. There are any number of parabolas that can be drawn through *C* and *B*, but the most suitable and convenient curve in this case is that which will have *AB* as axis, and therefore *B* and *A* for the vertices.

Two methods of obtaining the curve are shown, one on each side of the centre line *HC*.

First Method.—Join *CB* and find the mid-point *k*. Draw *kK* parallel to *AB* to meet a perpendicular to *AB* from *B*. Bisect *kK* in *q* and draw *nql* parallel to *CB*. Draw *qB*, *qC*, and parallels to *AB* from *n* and *l* to intersect them in *m* and *o*. Bisect *on* and *ml* in points *p* and *r*. Then a curve through *C*, *r*, *q*, *p*, *B* tangent to *CK*, *nl*, and *KB* is the required parabola. In a full-scale drawing tangents at *p* and *r*, parallel to *qB* and *qC*, would be helpful in drawing the curve.

Note.—Since *k* is the mid-point of the chord *CB* and *q* is the point of contact of a tangent drawn parallel to *CB*, the straight line through *q* and *k* is a diameter, and is therefore parallel to the axis ;

and since KB, the tangent at B, is perpendicular to kq, B is the vertex and BH, being parallel to kq, is the axis of the parabola.

The second method of constructing the curve is shown to the left of CH, and is as follows: Draw AK^1 perpendicular to AB and equal to half HC. Join K^1C. Divide K^1C and K^1A into the same number of equal parts and join the points of division in reverse order, as shown, to obtain the tangents ss^1, tt^1, uu^1. A flexible rod held to touch all the tangent lines will give a very accurate curve when setting out the full-sized drawing.

To obtain the joint lines, which are of course normals to the curve, determine the sub-normal HG by drawing CG perpendicular to CK^1. Make AF equal to half HG; then F is the focus. Divide AC to suit

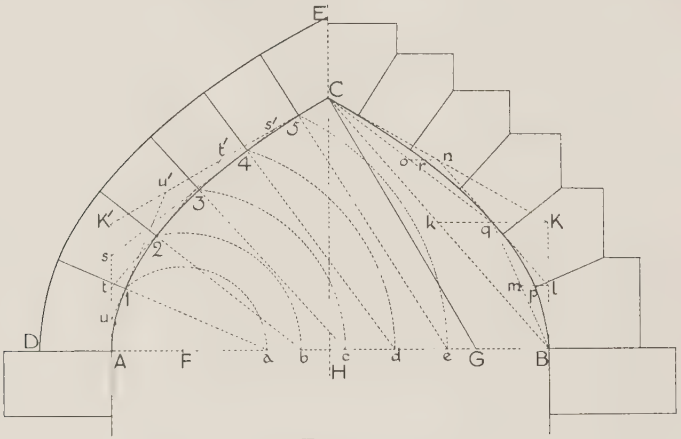


Fig. 82.

the number and positions of the voussoirs so as to obtain points 1, 2, 3, 4, 5. With F as centre, $F1$, $F2$, $F3$, $F4$, and $F5$ as radii, cut AB in points a, b, c, d, and e. Then $a1$, $b2$, $c3$, $d4$, and $e5$ are the directions of the joint lines and the parallel curve DE may be determined by making the joint lines all of equal length. The joint lines on the right of HC are also equal in length, and are determined by projecting horizontally from the other side. The stepped outline is obtained by drawing parallels and perpendiculars to AB from the extremities of the joint lines.

THE HYPERBOLA

The hyperbola was defined early in this chapter as the locus of a point moving in a plane in such a manner as to have the ratio of its distances from a fixed point and a fixed straight line in the plane constant and greater than unity; the fixed point being the focus, and

the fixed line the directrix. In Fig. 83 some important properties of the curve are illustrated. AA^1 is the transverse axis and BB^1 is the conjugate axis. The hyperbola is symmetrical about both axes, and C, the point of intersection of the axes, is the centre. The axes bisect each other perpendicularly. The points A and A^1 are the vertices, F and F^1 the foci, and $AF = A^1F^1$. DD and D^1D^1 are the directrices and extend indefinitely in both directions perpendicular to AA^1. A circle drawn with AA^1 as a diameter cuts the directrices in points h, h, h^1, h^1 such that the angles ChF, Ch^1F^1 are right angles, and straight lines through Chh^1 are asymptotes to the curve. Tangents to the curve at A and A^1 intersect the asymptotes in points E and E^1 such that CE

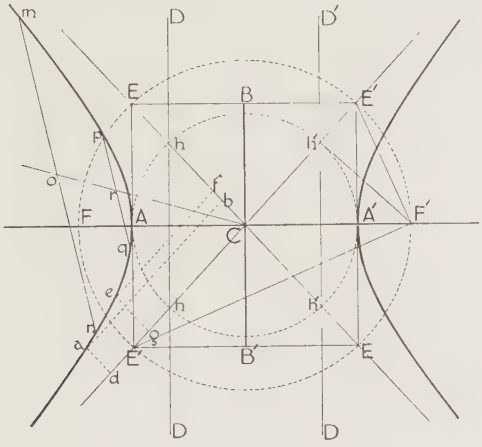


Fig. 83.

and CE^1 are each equal to CF. The conjugate axis BB^1 is equal to twice AE. The common bisector of any two parallel chords passes through the centre; thus, mn and pq are parallel chords, o and r are their mid-points, then the common bisector, or, produced, passes through C.

Parallelograms having two sides in the asymptotes and the opposite sides intersecting on the curve have a constant area; thus, the areas of the parallelograms $abCd$, $efCg$ are equal. If any tangent be drawn to intersect both the vertex tangents, the part lying between the points of intersection will subtend a right angle of either focus; thus the asymptote is a tangent, its point of contact being at infinity, and E^1, E^1 are the points of intersection with the vertex tangents; then $E^1F^1E^1$ is a right angle, and any other tangent to the curve is similarly related.

Properties of Tangents to the Hyperbola.

The difference between the focal distances of any point in the curve is constant and equal to the transverse axis.

Illustration. *Fig.* 84.—Let F, F^1 be the foci and P be any point in the curve; then $PF^1 - PF = AA^1$.

The tangent at any point in the curve bisects the focal distances of the point.

Illustration. *Fig.* 84.—The tangent at P bisects the angle FPF^1.

The feet of perpendiculars to any tangent from the foci are in the circumference of the circle on the transverse axis.

Illustration. *Fig.* 84.—Let Fe be perpendicular to the tangent Pe, then e is in the circumference of the circle, centre C, radius CA. Also P^2bF^1 and P^2dF are right angles.

Difference Circles. *Fig.* 84.—Let two circles DS and D^1S^1 be drawn about the foci as centres, with radii equal to the difference in the focal distances, that is, equal to AA^1, and let them be called "the difference circles." Then the longer of any pair of focal distances is cut by one or the other difference circle into two segments, such that the shorter focal distance is equal to the adjacent segment, and the straight line joining the point of intersection to the second focus is bisected by the circle on the transverse axis and also by the tangent at the point on the curve from which the focal distances are drawn.

Illustration. *Fig.* 84.—The focal distance F^1P is cut by the difference circle D^1S^1 in point c; then $cP = FP$, and the straight line Fc is bisected by the circle through AA^1 in e, and the tangent at P passes through e.

If straight lines be drawn from the foci to a point of intersection between a difference circle and the focal circle, one of the lines will be a tangent to the circle on the transverse axis, and the other line will be a tangent to the circle on the conjugate axis, the mid-points of the lines being the points of contact. Also the asymptote will be parallel to one of the lines and will bisect the other perpendicularly.

Illustration. *Fig.* 84.—The difference circle D^1S^1 intersects the circle through F and F^1 in the point R; then FR is tangent to the circle AA^1 at t, and RF^1 is tangent to the circle BB^1 at s, t and s being the mid-points of RF and RF^1. Also the asymptote TT^1 passes through t, is parallel to RF^1, and is perpendicular to RF. FRF^1 is a right angle, being in the semicircle on FF^1. If the asymptote is considered as the limiting position of a tangent to the curve, and F^1R as part of the corresponding focal distance, the relationship between

TT^1, RF^1, and RF is the same as with other tangents and corresponding lines. NOTE.—RF^1 and TT^1 are parallel to a line through A^1B.

The tangent at the point of intersection of the curve and a difference circle passes through the point of intersection of the same difference circle and the circle on the transverse axis, and a focal distance of the point of contact passes through the point of intersection of the circle on the transverse axis and the second difference circle.

Illustration. *Fig.* 84.—P^2 is the point of intersection of the curve and a difference circle; therefore the tangent at P^2 passes through d and the focal distance FP^2 passes through a.

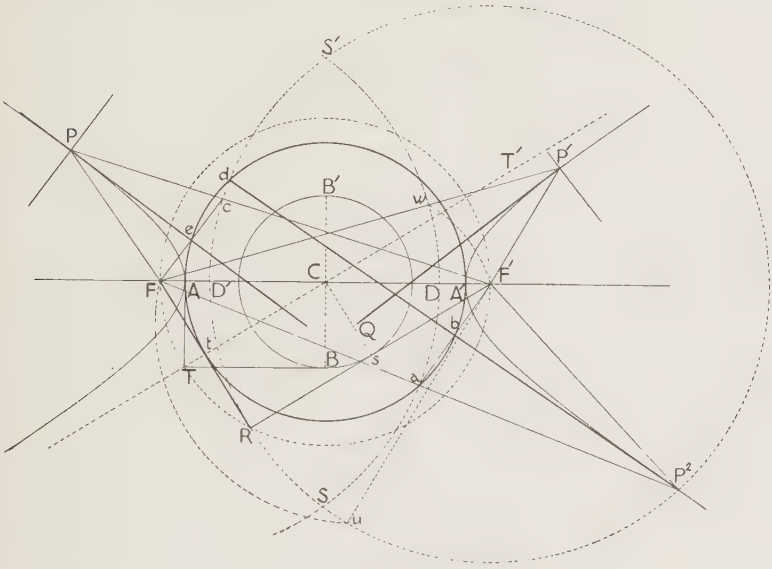


FIG. 84.

Problem 35. *Fig.* 84.—**From any given point outside the curve, to draw a tangent and determine the point of contact.**

Let Q be the given point outside the curve. With Q as centre, QF^1 as radius, cut the difference circle in w. Through F and w draw a straight line to meet the curve in P^1; then QP^1 is the required tangent and P^1 is the point of contact. A second tangent may be drawn from Q by taking QF as radius, Q as centre, and cutting the other difference circle; then a point of contact may be found, as before, on the other branch of the curve. NOTE.—If with Q as centre, radius QF, the difference circle be cut in u, as shown in the figure, a straight line through uF^1 will meet the curve in P^1 and the tangent P^1Q will, if produced, bisect the arc, or distance, Fu.

Construction of Hyperbola from Minimum Data.

Problem 36. *Fig.* 85.—**Given the foci and either the transverse or conjugate axis, to determine the second axis, the directrix, the asymptotes, and to obtain points in the curve by intersecting lines.**

Let F, F^1 be the given foci and BB^1 the given axis. Draw a straight line through F and F^1 to intersect BB^1 in C; then C is the centre. On BB^1 draw the circle, centre C, and from F^1 draw a tangent to touch it in b and extend to c, making bc equal to bF^1. Join cF and draw Ca parallel to cF^1; then Ca produced is an asymptote and is the perpendicular bisector of cF. Through a, draw DD^1 perpendicular to FF^1, and with C as centre, Ca as radius, cut FF^1 in A and A^1; then DD^1 is the directrix and AA^1 is the transverse axis. When the two axes are given instead of one axis and the foci the same construction may be used, if F^1c and Ca are drawn parallel to A^1B.

To determine the curve by intersecting lines. First plot any point E such that $F^1E - FE = AA^1$; then E is a point in the curve. On EA as diagonal construct the rectangle $AHEG$ and divide EG and EH into a corresponding number of equal parts and number as shown. Then straight lines from A to the points in EH, and from A^1 to the points in EG, intersect in points on the curve, indicated at x, y, and z.

Problem 37. *Fig.* 85.—**Given (1) an asymptote and the conjugate axis, or (2) an asymptote and the transverse axis, or (3) an asymptote and the foci, or (4) an asymptote, a directrix, and the centre, to determine any other points or lines related to the curve.**

(1) Let CT be the given asymptote and BB^1 the given conjugate axis.

With centre C draw the auxiliary circle BB^1, and tangent to it draw cF^1 parallel to CT. From B draw BA^1 parallel to CT; then F^1 is a focus and A^1 a vertex. Make CF equal to CF^1 and CA equal to CA^1. Draw Fa perpendicular to CT and draw DD^1 through a parallel to BB^1.

(2) Given CT and AA^1. With centre C draw the auxiliary circle AA^1 cutting CT in a. Through a draw Fc perpendicular to CT. Make ac equal to aF, F being in AA^1 produced. From c draw cF^1 parallel to CT. Draw Cb perpendicular to CT and make CB and CB^1 each equal to Cb and perpendicular to AA^1. Draw DD^1 as before.

(3) Given CT, F, and F^1. Draw Fc perpendicular to CT and cutting it in a. Make ca equal to aF, and proceed as before.

(4) Given C, CT, and DD^1. Draw a perpendicular to CT through a and make the angle aCF equal to the angle D^1aF. Make CA equal to Ca, and proceed as before.

Alternative solution. With centre C, radius Ca, draw the auxiliary circle, and from C perpendicular to DD^1 draw a line to cut it in A, and proceed as before.

Problem 38. *Fig.* 86. –**Given only the curve of a hyperbola, to determine the axes, foci, directrices, and asymptotes.**

Let *mpnu* be the given curve.

Draw any two pairs of parallel chords *po*, *mn* and *st*, *vu*, and bisect them in points *q*, *r*, *g*, and *h*. Draw straight lines through *r*, *q* and *g*, *h* to intersect in a point *C*; then *C* is the centre of the conic. With *C* as centre, and any convenient radius, cut the curve in points *G*

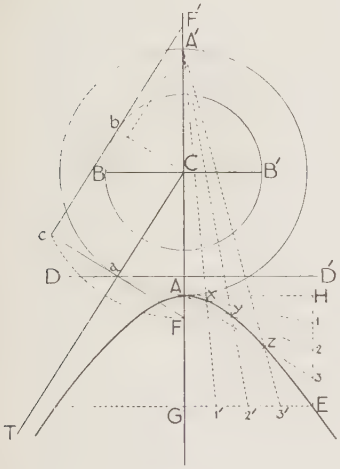


Fig. 85.

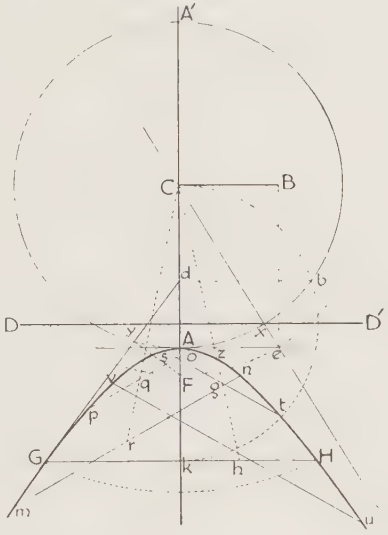


Fig. 86.

and *H*. Draw *GH* and bisect it in *k*; then a straight line through *C* and *k* is the axis and will cut the curve in *A*, the vertex. Make CA^1 equal to *CA*; then AA^1 is the transverse axis. Draw the auxiliary circle through AA^1 and draw any tangent to the given curve to cut it in *y*. A tangent may be drawn accurately in either of the following ways.

The common bisector of the chords, *st*, *uv*, cuts the curve in a point *z* and the tangent at *z* is parallel to the chords. Or, let *G* be any point on the curve and *Gd* a tangent at *G*; then the ordinate from *G* meets the axis in a point *k*, such that $Cd \times Ck = CA^2$. Then *d* may be determined as follows: Draw a semicircle on *Ck* cutting the auxiliary circle in *b*; then *Cb* is the mean proportional between

Ck, *Cd*. From *b* drop a perpendicular on to the axis at *d*; then the tangent at *G* passes through *d*. A perpendicular to *Gd* from *y* will cut the axis in *F*; then *F* is the focus. Draw the vertex tangent *Ae*, and with *C* as centre, *CF* as radius, cut it in *e*; then the asymptote is a straight line through *Ce*. Through *x*, the point of intersection between *Ce* and the auxiliary circle, draw the directrix DD^1 perpendicular to AA^1. *CB*, the semi-conjugate axis, is determined by drawing *eB* parallel to AA^1, *CB* being perpendicular to AA^1.

Problem 39. *Fig.* 87.—**To determine points in a hyperbola, having given the asymptotes and a point in the curve.**

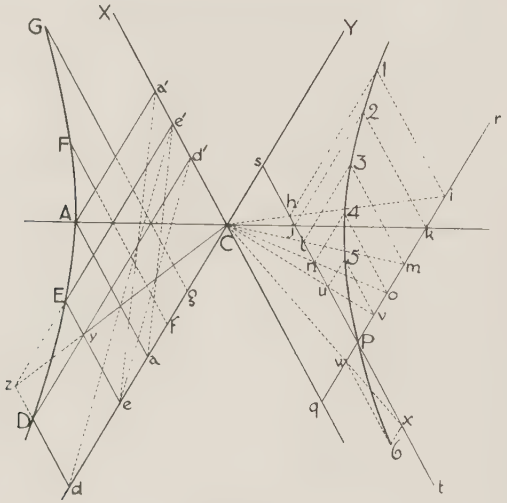


Fig. 87.

(1) Let *CX*, *CY* be the given asymptotes and *P* the given point.

Through *P* draw *qr* and *st* parallel to *CY* and *CX*, and from *C* draw straight lines to intersect them in any points, as *h*, *i*; *j*, *k*; *l*, *m*; . . . *w*, *x*. From the points of intersection of each radial line with *st* and *qr* draw parallels to *CY* and *CX* to intersect in points 1, 2, 3, . . . 6; then these are the required points in the curve.

(2) Let *A*, *E*, or *D* be the given point.

Construct any number of parallelograms having a constant area, each with a pair of adjacent sides, one in each asymptote; then the point of intersection of the other two sides is in the curve. Thus the parallelograms $AaCa^1$, $EeCe^1$, $DdCd^1$ are of equal area; then the curve through *A*, *E*, *D* is a hyperbola. The construction shown in the figure is as follows: Let *A* be the given point. Draw *Aa*, Aa^1 parallel

to CX and CY. From any point e^1 draw e^1E parallel to Aa^1 and draw a^1e parallel to e^1a; then a line from e parallel to Aa will intersect the line from e^1 in E, and the area $EeCe^1$ equals the area $AaCa^1$. Any number of points may be found in the same way, and corresponding points on the opposite side of CA may be determined from them by making $Cf = Ce^1$, $Cg = Cd^1$, $gG = d^1D$, and $fF = e^1E$. To prove that the areas $AaCa^1$, $EeCe^1$ are equal: join a, a^1 and e, e^1; then because the triangles aee^1, aa^1e^1 have the common base ae^1 and are between the same parallels ae^1, a^1e, they are equal in area. Add the triangle ae^1C to each triangle; then the areas aa^1C, ee^1C are equal, and because aa^1C equals half $AaCa^1$ and ee^1C equals half $EeCe^1$, the two parallelograms are equal.

A second proof, applicable to the construction on the right of the figure also, is as follows: Through any point y in the line Ee draw a straight line from C to meet e^1E, produced in z. From z, parallel to Ee, draw zd, and through y, parallel to Ee^1, draw d^1D; then the parallelograms $EeCe^1$, $DdCd^1$ are equal. For, since Cz is a diagonal of the whole parallelogram $Czde^1$ and yz is a diagonal of the parallelogram $yzDE$, the triangles Cdz, Ce^1z are equal, and the triangles Dzy, Ezy are equal. Take Eyz from Cze^1 and Dyz from Czd, then the remainders, $EyCe^1$, $DyCd$, are equal, and because the triangles Cey, Cd^1y are equal, the parallelograms $EeCe^1$, $DdCd^1$ are equal. It will be seen that the two constructions shown in the figure are based upon the same property of the curve, all the parallelograms on the right of the figure being related to the curve precisely as is $DzEy$ on the left.

Conics as Sections of the Cone

THE ELLIPSE

The section of a right circular cone made by a plane inclined to the axis of the cone so as to cut all the generators on one side of the vertex is an ellipse.

Illustration. *Fig.* 88.— V is the vertex of the cone, VO is the axis, and Vr, Vs the outline generators. Let AA^1 be an edge view of any plane cutting all the generators on the same side of V. Then when AA^1 is at right angles to the axis VO, the section is a circle, and when at other than right angles, an ellipse. In the figure, AA^1 is not at right angles to VO, and the section is an ellipse, having AA^1 as the transverse or major axis, and BB^1, the perpendicular bisector of AA^1, as the minor axis. C is the centre. Let ac be perpendicular to VO through C intersecting VO in Q, and let abc be the rabatment of half the circular section aQc; then, if Cb is perpendicular to ac, the distance through the cone at the position C, measured perpendicular to the

axial plane *Vsr*, is twice *Cb* and is equal to the minor axis, therefore *CB* and CB^1 are each equal to *Cb*.

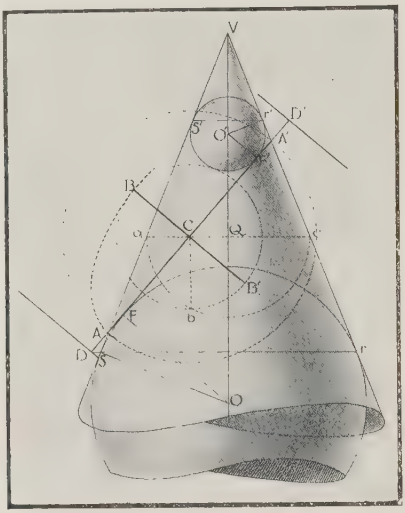


Fig. 88.

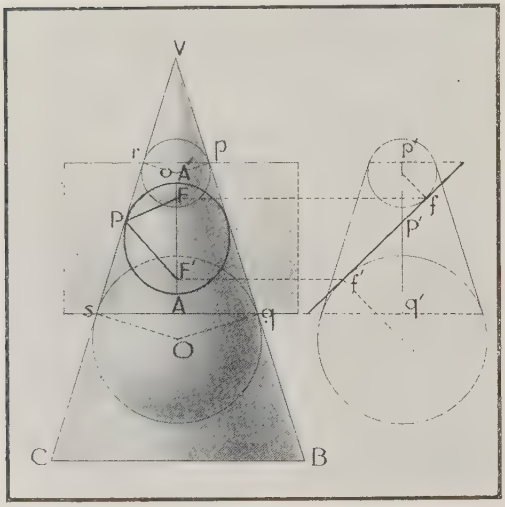


Fig. 89.

A sphere inscribed in a cone and touching a section plane of the cone is called a focal sphere, and the point of contact between the sphere and the plane is a focus of the section made by the plane.

Illustration. *Fig.* 88.—Let O and O^1 be the centres of two spheres touching the cone in circles of contact at rs and r^1s^1 and touching the section plane AA^1 in points F and F^1; then the spheres are called focal spheres and the points F and F^1 are the foci of the elliptic section, and $AF = A^1F^1$.

The major (or focal) axis AA^1 is equal to the distance between the contact circles of the cone and spheres measured along the surface of the cone; thus $ss^1 = AA^1$. This is shown more clearly in Fig. 89, in which rp and sq are the circles of contact of the focal spheres O, o with the cone VCB, and F, F^1 are foci of the elliptic section A, A^1. Let P be the point in which the section plane cuts the generator VC, then PF, PF^1 are focal distances which together equal the major axis AA^1; and since F and F^1 are the points of contact between the section plane and the spheres, the straight lines PF, PF^1 are tangents to the spheres from the point P; but Pr, Ps are also straight line tangents to the spheres from P, and because tangents from a common point to a common sphere are equal in length, $PF = Pr$ and $PF^1 = Ps$; then $AA^1 = sr$. This demonstrates also that the sum of the focal distances is constant and equal to the major axis; for any other point in the outline of the section may be similarly dealt with.

The plane containing the circle of contact between the cone and the focal sphere intersects the section plane in the directrix of the section. In the ellipse, since there are two focal spheres, there are two directrices.

Illustration. *Fig.* 88.—Planes perpendicular to the axis of the cone and containing the circles of contact rs, r^1s^1 intersect the section plane at D and D^1, and straight lines on the section plane, perpendicular to the major axis, are the directrices of the ellipse. The directrices, minor axis, and part of the curve are shown rabatted into the axial plane.

THE PARABOLA AND CONE

The section of a cone by a plane parallel to one generator only of the cone is a parabola.

Illustration. *Fig.* 90.—Let the straight line AC^1 be the edge view of a section plane to which the generator EV is parallel, then the section is a parabola, and since EV is in the axial plane, that is, in the plane containing the axis of the cone and perpendicular to the section plane, no second generator is parallel to the section plane.

Note.—There are two generators parallel to a hyperbolic section plane, and no generators are parallel to an elliptic section plane.

Only one sphere can be inscribed in a cone to touch a parabolic section plane, therefore a parabola has only one focus and one directrix,

and because the inclination of the section plane is equal to the base angle of the cone, the curve cuts the axis at the mid-point between the focus and the directrix, and all other points in the curve are equidistant from the focus and the directrix.

Illustration. Figs. 90 and 91.—In Fig. 90, C is the centre of the focal sphere, F is the point of contact between the section plane and the sphere, and dd^1 is an edge view of the circle of contact between the cone and the sphere. The plane containing the circle of contact dd^1 intersects the section plane in a line through D, perpendicular to the axial plane; this line is the directrix, and is shown, with part of the curve, rabatted into the axial plane at DD^1. F is the focus, A the vertex, and FP the semi-latus rectum, and $FP = FD$, and $FA = DA$. For since C^1D is parallel to EV, the angles EVC^1, VC^1A, C^1VA are equal, and because dD is parallel to EC^1, the angles DdA, dDA are equal, and $AD = Ad$; but $Ad = AF$, being tangents from A to the same circle; then $DA = FA$. To show that all points on the curve are equidistant from the focus and the directrix (Fig. 91). Let P be any point in the curve CAB, and DQ the line of intersection between the section plane and the plane containing the circle of contact between the cone and focal sphere; then DQ is the directrix. Let F be the focus. Join PF. Draw Ps in the direction PV, s being in the contact circle, and draw Pr perpendicular to DQ; then PF is a tangent to the sphere at F and Ps is a tangent to the sphere at s; therefore $PF = Ps$. Since Ps is part of a generator of the cone, and the inclination of the section plane is equal to the base angle of the cone, and Pr is a line in the section plane perpendicular to DQ, Pr and Ps are equally inclined, and because they reach the same altitude above P they are equal in length. Then because $Ps = PF$, and $Ps = Pr$, $PF = Pr$ and all points in the curve are equidistant from the focus and the directrix. The student should consider the construction lines employed for rabatting the curve in Fig. 90.

THE HYPERBOLA AND CONE

The section of a cone by any plane which cuts both branches (or sheets) of the cone is a hyperbola.

Illustration. Fig. 92.—Let the double cone, vertex V, be cut by the section plane SS^1, then because the plane intersects the cone on each side of V, the section is a hyperbola, and because a focal sphere may be inscribed in each branch of the cone to touch the section plane, the conic will have two directrices and two foci. The points AA^1 in which the section plane intersects the generators contained by the

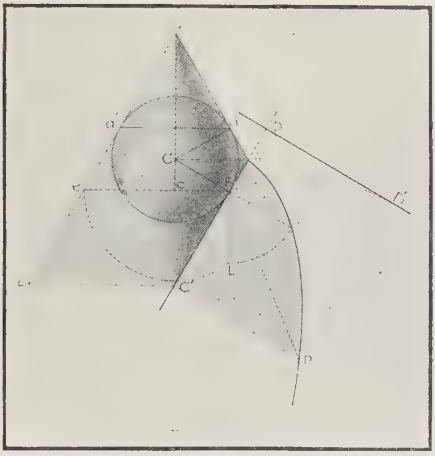


FIG. 90.

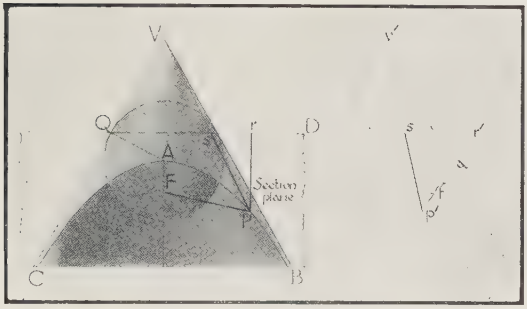


FIG. 91.

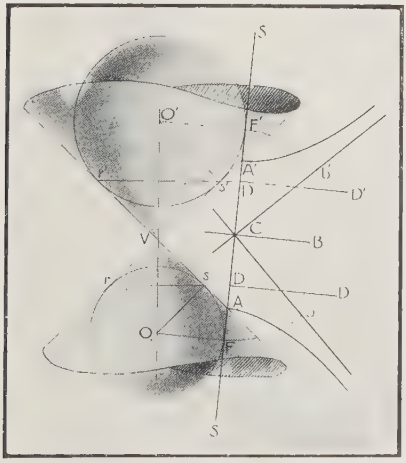


FIG. 92.

axial plane are the vertices, and the straight line joining AA^1 is the transverse axis. C, the mid-point of AA^1, is the centre of the conic, and CB, perpendicular to AA^1, is the semi-conjugate axis (rabatted). F and F^1, the points of contact between the section plane and the focal spheres, are the foci, and the planes containing the circles of contact rs, r^1s^1 intersect the section plane in the directrices D, D^1 (DD, D^1D^1 are the directrices rabatted into the axial plane). The asymptotes Cb, Cb^1 (rabatted) are the straight lines in the section plane through C parallel to the two generators of the cone contained by a plane through the vertex parallel to the section plane. Further relationships between the rabatted curves and lines have already been discussed.

The transverse axis is equal to the distance between the circles of contact between the cone and focal spheres measured along a generator of the cone.

Illustration. *Fig.* 93.—Let CAB, $C^1A^1B^1$ be the sections of the two sheets of the double cone, vertex V, by the plane containing the

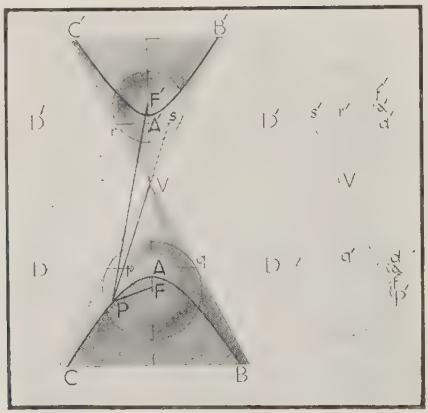


Fig. 93.

directrices DD, D^1D^1, and let the focal spheres touch the plane in the focal points F, F^1 and the cone in circles of contact containing p, q, r, and s. Let P be any point in the hyperbolic outline of the section. Join PF, PF^1 and draw a straight line through P and V to meet the contact circles in points p and s. Then $PF^1 = Ps$, being tangents to the same sphere from P, and $PF = Pp$, both being tangents to the other sphere from P; therefore $PF^1 - PF = Ps - Pp$, and because ps lies in the surface of the cone with an extremity in each contact circle, $ps = qr$. Since P is any point in the curve, the vertex A may be considered in the same way; then $AF^1 - AF = ps = rq$; but $AF^1 - AF = AA^1$; therefore, AA^1 equals rq, the distance between the contact circles measured along a generator. This demonstrates also that the difference between the focal distances is constant and equal to the transverse axis.

SECTION II

DESCRIPTIVE GEOMETRY

CHAPTER VII

THE STRAIGHT LINE

Introduction.—The student is assumed to be acquainted with the fundamental rules of projection and to be capable of dealing with simple problems relating to points and lines in space; nevertheless, it is considered advisable to give the fullest explanations of the methods employed in the work contained in the following pages, and fundamental rules will be reiterated for the purpose of impressing them on the student's mind. Problems that appear involved and difficult are usually nothing more than a combination of many fundamental constructions, and the chief cause of failure in attempts at such problems is inability to recognise the types of fundamentals required in the solutions. This weakness tends to disappear as the student's mind develops a more intimate acquaintance with the simple but highly important principles that recur with nearly every new application. Briefly, a rule properly assimilated rarely requires finding; it presents itself to the mind when needed.

Most of the difficulties met with in descriptive geometry arise out of the limitations imposed by a flat surface for depicting and manipulating lines in space, but if sufficient time is devoted to the consideration of one or two characteristic positions of the straight line in space and its relationship with the co-ordinate planes, nearly all the difficulties will disappear.

The projection of a line on any plane is the straight line joining the feet of perpendiculars to the plane from the extremities of the line.

Illustration. Fig. 94.—Let XY be the line of intersection of the planes of projection shown in oblique projection, and let AB be a straight line inclined to both planes; then if Aa and Bb are perpendicular to HP (the horizontal plane), ab is a projection of AB on the plane and is called the plan of the line. Similarly, if Aa^1 and Bb^1 are perpendicular to VP (the vertical plane), a^1b^1 is the projection of AB on that plane and is the elevation of the line.

Let it be understood that all planes are unlimited in extent and that any straight line may be produced indefinitely in both directions. Then any straight line produced indefinitely will intersect all the planes, other than those to which the line is parallel, and the point of intersection between the line and a plane is called the trace of the line on the plane. In Fig. 94 the line AB is inclined to both planes and is shown produced to meet the two planes in $HT(m)$ and $VT(n^1)$; then HT is the horizontal trace and VT the vertical trace of the line. The projection of a line, produced if necessary, passes through the trace situated in the plane containing the projection; thus, a^1b^1 produced passes through VT, and ab produced passes through HT. For, since a^1A and b^1B are parallel lines, both being perpendicular to VP, the projection a^1b^1 and the line AB lie in an inclined plane whose line of intersection with VP contains a^1b^1; then AB produced must meet VP in the line of intersection, between VP and the plane containing a^1b^1 and AB, and since a^1b^1 is in the line of intersection, AB produced meets a^1b^1 produced in VT. Similarly AB and ab lie in a vertical plane whose horizontal trace contains ab; then AB produced meets ab produced in the horizontal trace of AB.

The inclination of a line to a plane is equal to the angle between the line and its projection on the plane.

Illustration. *Fig.* 94.—The angle between the line AB and its projection ab is marked θ, and since ab is in HP, the inclination of AB to the horizontal plane is θ. Also AB is inclined to the vertical plane at an angle equal to ϕ, since ϕ is the angle between AB and its projection a^1b^1.

When a line is parallel to a plane, the length of its projection on the plane is equal to the length of the line; when the line is not parallel to the plane containing the projection, the length of the projection is less than the length of the line.

Illustration. *Fig.* 94.—The straight lines joining the extremities of the line to the extremities of its projections are called projectors. Consider the projectors Aa^1, Bb^1. If AB were parallel to VP, Aa^1, Bb^1 would be of equal length; and since both are perpendicular to VP, the line AB, its projection a^1b^1, and its projectors Aa^1, Bb^1 would form a rectangle of which the sides AB, a^1b^1 would be opposite and, therefore, equal.

In Fig. 94 the line AB is inclined to both planes of projection; therefore its real length is greater than the length of either projection. For since Bb is perpendicular to AB, and AB produced meets ab produced in point m (the horizontal trace), mbB is a right-angled triangle having mB as hypotenuse and mb as base; therefore mB

is greater than mb; and since aA is parallel to the perpendicular bB, $AB : ab = mB : mb$; then AB is greater than ab. If the right-angled

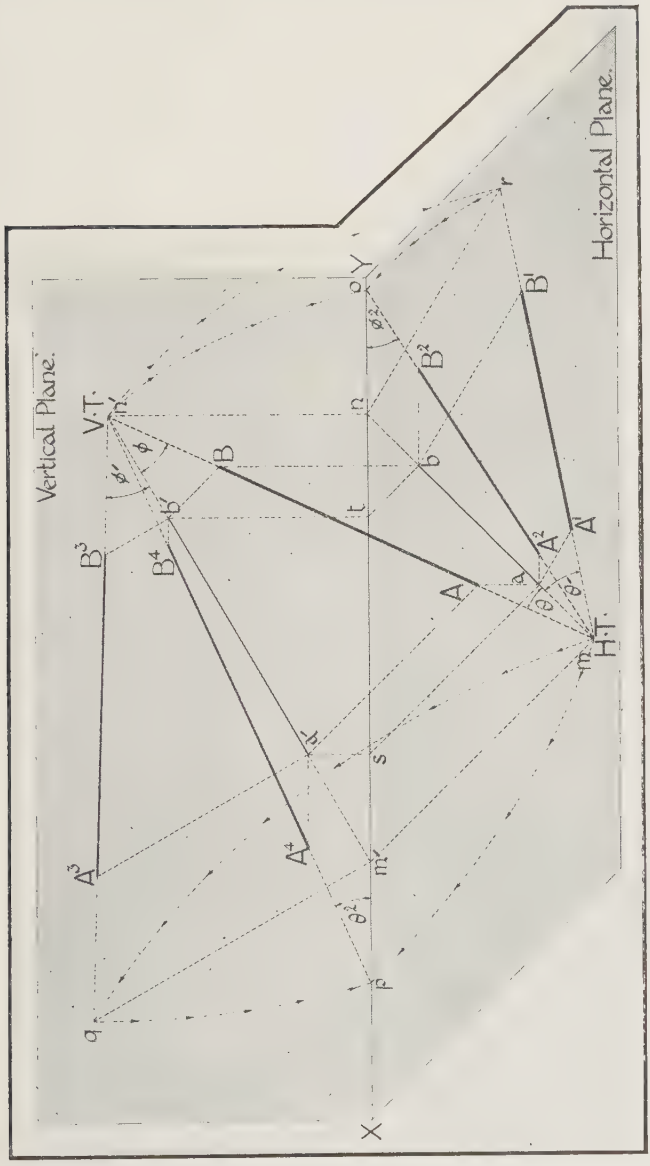


Fig. 94.*

triangle Aa^1n^1 is considered in the same way, the projection a^1b^1 may be shown to be less than the line AB.

The true length and inclination of any oblique line may be determined by finding the true shape of the quadrilateral formed by the line, its projection, and the two projectors. The most convenient method of finding the true shape is that of rabatment about the projection into the plane containing the projection.

Illustration. *Fig.* 94.—Let aA^1 be perpendicular to ab and equal to sa^1, and let bB^1 be perpendicular to ab and equal to tb^1; then, because $sa^1 = aA$ and $tb^1 = bB$, and the angles Aab, Bba are right angles, the two quadrilaterals $aABb$, aA^1B^1b are congruent, and $A^1B^1 = AB$, and the angle between A^1B^1 and ab is equal to the inclination of the line AB to the horizontal plane. Also, if a^1A^3 and b^1B^3 are perpendicular to a^1b^1 and equal to sa and tb respectively, the quadrilaterals a^1ABb^1, $a^1A^3b^1B^3$ are congruent, and A^3B^3 is equal to the true length of AB, and the angle between A^3B^3 and a^1b^1 is equal to the angle between the line AB and the vertical plane.

Other methods of finding the length and inclinations of a line shown by its projections, involving the use of the traces, are shown in the figure and are as follows: Let n be the plan of the vertical trace and m^1 the elevation of the horizontal trace; then mnn^1 is a right-angled triangle, whose plane is perpendicular to HP and whose elevation is m^1n^1n; nn^1 being in VP. If the triangle is turned about nn^1, the point m will describe a circular arc, centre n, on HP which will cut the ground line in a point p, such that the triangles pnn^1, mnn^1 are congruent and the angle npn^1, marked θ^2, equals the inclination of AB to HP. Parallels to pn from a^1 and b^1 cut pn^1 in A^4, B^4. Then A^4B^4 is the true length of the line AB. In the same way the right-angled triangle mm^1n^1, whose plan is mm^1n, is turned about mm^1, n^1 describing a circular arc, centre m^1, in the vertical plane, meeting the ground line in o; then the angle m^1om is the angle made by the line to VP, and parallels to m^1o from a and b meet mo in A^2, B^2. A^2B^2 is the real length of AB.

Note.—In each case the rabatment of AB produced passes through HT or VT, and a perpendicular to ab from n meets A^1B^1 produced, in r, such that $nr = nn^1$; $mr = mn^1 = mo =$ the distance between the traces. Also, if m^1q^1 is perpendicular to m^1n^1, q being in A^3B^3 produced, $m^1m = mq$, and qn^1 equals the distance between the traces.

It should be understood that Fig. 94 is only a pictorial representation of the line, projections, and rabatments, and that all the distances and angles are distorted.

The student should now compare Figs. 94 and 95, the latter showing the accurate geometrical method of obtaining the lines and angles in a case corresponding to that illustrated in the former

figure. In Fig. 95 ab is the plan, a^1b^1 the elevation of a straight line, the position of the line relative to the planes of projection being similar to that of AB, Fig. 94. The letters in the two diagrams indicate corresponding lines and angles, the centres of the circular arcs and the positions of equal lengths and equal angles in Fig. 95 being as follows : The true length of the line whose projections are ab, a^1b^1 is shown at A^1B^1, A^2B^2, A^3B^3, and A^4B^4. $aA^1 = sa^1$, $bB^1 = tb^1$, $a^1A^3 = sa$, $b^1B^3 = tb$. The real distance between the traces VT and HT is equal to the length of each of the lines mr, mo, pn^1, qn^1. The inclination of the line to HP is shown twice at θ^1 and θ^2 ; the inclination to VP is ϕ^1 and ϕ^2.

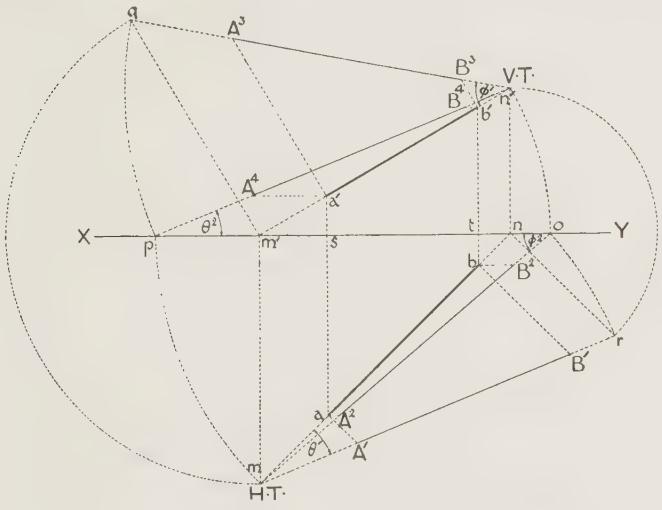


Fig. 95.

The four lines aa^1, bb^1, mm^1, nn^1 are each perpendicular to XY, and all the following angles are right angles : A^1ab, B^1ba, mnr, qm^1n^1, $A^3a^1b^1$, $B^3b^1a^1$. Parallels to XY are A^4a^1, B^4b^1, A^2a, B^2b. The centres of the arcs are : arc ro, centre m ; arc n^1o, centre m^1 ; arc mp, centre n ; arc pq, centre n^1 ; arc qm, centre m^1 ; arc n^1r, centre n.

In practical applications of descriptive geometry points and lines are arranged, as far as possible, in the first dihedral angle ; that is, above the horizontal plane and in front of the vertical plane. It is not always convenient and possible, however, to maintain this condition, since, in the first place, it is difficult to foresee in what directions and to what extent a problem may develop, and secondly, the available drawing surface may render economy of space necessary and overlapping projections then become unavoidable. In Figs. 96 and 97 a straight

line which passes from the first into the second and fourth dihedral angles is illustrated and the construction for finding the length and inclinations shown.

First consider Fig. 96. AB is a straight line having the section lying between VT and HT in the first dihedral angle, the section between VT and B in the second dihedral angle, and the section between HT and A in the fourth dihedral angle, VT (n^1) being the point in which AB intersects the vertical plane, and HT (m) the point in which AB intersects the horizontal plane. a^1 and b^1 are the projections of A and B on to VP, and a, b the projections on to HP; then the straight line joining a and b is the plan and the straight line joining a^1b^1 is the elevation of AB. The elevation of HT is m^1, m^1 being the point of intersection between the elevation a^1b^1 and the ground line, and n, the plan of VT, is the point of intersection between the plan ab and the ground line. Let ap, bq, a^1p, and b^1q be each perpendicular to the ground line; then ap is the distance of A from, *and in front of*, the vertical plane; bq is the distance of B from, *and behind*, the vertical plane; a^1p is the distance of A *below* the horizontal plane; b^1q is the height of B *above* the horizontal plane.

Now suppose the horizontal plane to turn about the ground line in the direction indicated by the arrow-points on the arc aa^2 until the positions of the two planes coincide; then a will be at a^2, b will be at b^2, and qb^2, pa^2 will be equal to qb and pa respectively. The straight line joining a^2, b^2 will pass through n, since n is in the ground line, and a^2b^2 is the position of the plan of AB relative to the elevation a^1b^1 as required in orthographic projection. A perpendicular to the ground line from m^1 cuts a^2b^2 in ht, the rabatment of the horizontal trace. The inclination of AB to HP is the angle nmn^1; the inclination to VP is the angle m^1n^1m.

Now compare Figs. 96 and 97. In the latter figure the correct geometric method of determining the inclinations, true length, and traces of a line situated as in Fig. 96 is shown. Let ab, a^1b^1 be the given projections, the sections below HP and behind VP being dotted. Then if aA and bB are perpendicular to ab and equal to pa^1 and qb^1 respectively, AB is the true length of the line and θ is its inclination to HP. Also a^1A^1, b^1B^1 being perpendicular to a^1b^1 and equal to pa and qb, A^1B^1 is the true length, and ϕ the inclination to VP of the line whose projections are ab and a^1b^1. If from the point m, in which the elevation cuts the ground line, a line is drawn at right angles to XY to meet ab in HT, and a perpendicular to XY is drawn from n to meet a^1b^1 in VT, then HT and VT are the traces. On the right of the figure a second elevation of the line on a plane perpendicular

to XY is given, corresponding distances in the two figures being as follows: $q^2b^2=qb$; $q^1b^2=qb^1$; $ovt=n\,VT$; $oht=mHT$; $p^1a^2=pa^1$;

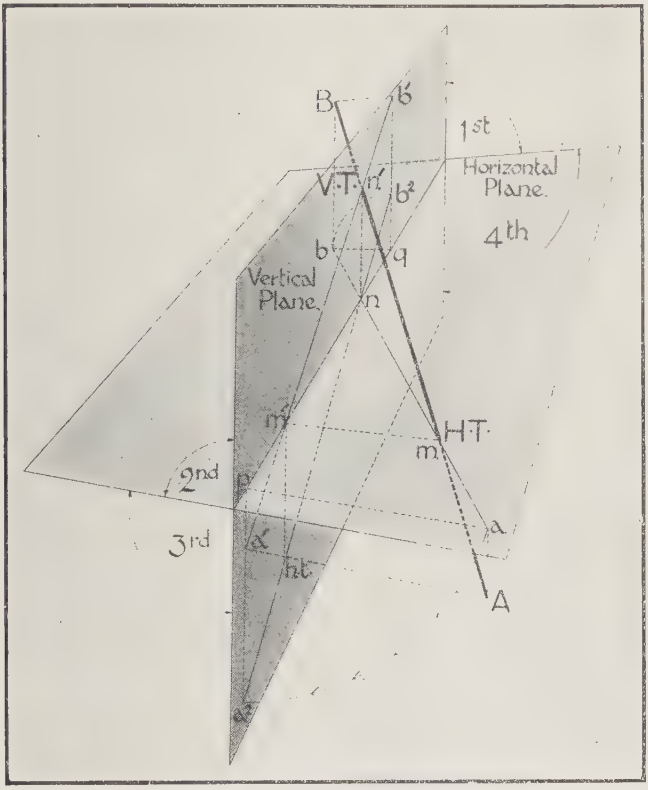


Fig. 96.

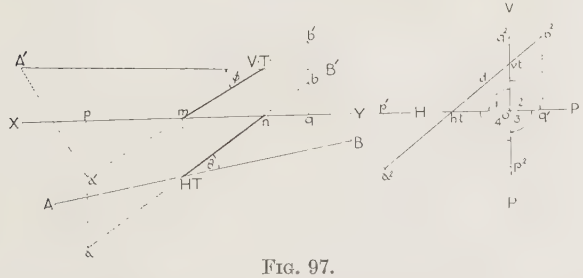


Fig. 97.

$p^2a^2=pa$. VP and HP are edge views of the co-ordinate planes and o is an end view of the ground line. The real distance of the line AB from the ground line is do, do being perpendicular to a^2b^2.

In Fig. 98 a second method of obtaining the length and inclinations of a straight line, positioned as in the last example, is shown. The method is that of turning the line about its traces into the planes containing the projections, as follows :

With m as centre, ma^1, mb^1 as radii, draw arcs to cut the ground line in points q and s. From q and s draw perpendiculars to XY to meet parallels to XY from a and b in A and B ; then AB is the true length of the line and makes an angle, ϕ, with the ground line equal to the inclination of the line to VP. Since m is the elevation of HT, AB will pass through HT. Next, with centre n, radii na and nb, cut the ground line in p and r. Draw pA^1 and rB^1 perpendicular to XY to meet a^1A^1, b^1B^1, drawn parallel to XY ; then A^1B^1 is the true

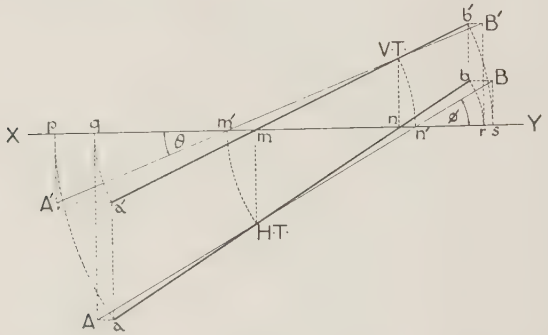


Fig. 98.

length and θ the inclination to HP of the line whose projections are ab, a^1b^1.

When the inclinations only are required they may be determined by turning VT and HT about m and n to n^1 and m^1 and joining n^1 to HT and m^1 to VT.

Changing the Ground Line.—The most important section of geometry required by the practical man, either in the workshop or on the actual building, is that which deals with the determination of true shapes of surfaces, the real angles between lines, the angles between lines and surfaces, and the angles between surfaces. For the greater number of cases met with in practice there are concise constructions, each specially suited to the nature of the problem to be solved ; these will, of course, be dealt with. Occasionally, however, a problem of a quite unusual nature is met with and a direct solution is difficult to find. Fortunately, the method of changing the ground line, which enables an object to be seen from any direction whatsoever, is based upon one or two of the most simple rules of projection, and is capable of supplying a solution

in practically every case. The work involved is sometimes excessive, but the chief consideration is to obtain the desired result.

In Fig. 99 several projections of the same straight line are shown, and if the figure is sufficiently understood the student will be enabled to turn any object into any desired position, and therefore the true shape of any of its surfaces or the magnitude of any of its angles may be determined.

For the purposes of projection any object may be considered as being made up of a number of lines and points, and since each line or point may be dealt with separately, no greater knowledge is required in dealing with a complicated shape than is required in the case of a single line. There is, of course, more work, and the risk of confusion is greater; but the latter is very much reduced if clear and careful lettering is resorted to and the order in which the lines are dealt with is a properly considered one. Referring to Fig. 99, let ab, a^1b^1 be the given projections of any straight line AB, then vt and ht are the vertical and horizontal traces. Now let X^1Y^1 be the second ground line, parallel to the plan ab; then, because (*a*) *the plan and elevation of a point are always in the same straight line drawn perpendicular to the ground line, and* (*b*) *the height of the projection of a point on to any vertical plane is equal to the height of the point above the horizontal plane*, if m^1a^3 equals ma^1 and n^1b^3 equals nb^1, and bb^3 and aa^3 are perpendicular to X^1Y^1, a^3b^3 is a second elevation of the straight line AB; and because the projection ab is parallel to X^1Y^1, the line AB is parallel to the vertical plane containing a^3b^3, therefore a^3b^3 is the true length of AB, and the angle between X^1Y^1 and a^3b^3 is equal to the inclination of AB to HP. Also, if a^3b^3 produced meets X^1Y^1 in d^1, then the line joining d^1 to ht is at right angles to X^1Y^1. Next, let X^3Y^3 be the ground line of *any* vertical plane; then by the foregoing rules, if aa^4 and bb^4 are perpendicular to X^3Y^3, and m^3a^4, n^3b^4 are equal to ma^1 and nb^1 respectively, a^4b^4 is a third elevation of AB, and a^4b^4 produced meets X^3Y^3 in a point d, such that the straight line joining d to ht is perpendicular to X^3Y^3.

Any projection from a plan is an elevation; any projection from an elevation is a plan. Therefore, because a^1b^1, a^3b^3, and a^4b^4 are projections from the plan ab, they are elevations; and because ab and a^2b^2 are projections from the elevation a^1b^1, they are plans.

Although a line or point may be projected on to any plane without reference to the co-ordinate planes, the expression "changing the ground line" is usually taken to mean the arranging of a plane perpendicular to one or the other of the co-ordinate planes; when the plane is perpendicular to HP, the projection on that plane is an elevation; when

the plane is perpendicular to VP, the projection on to it is called a plan. An example of the latter is shown at a^2b^2, where X^2Y^2 is any ground line, and the projection is made from the elevation a^1b^1; the distances m^2a^2, n^2b^2 being equal to ma and nb. If a^2b^2 be produced to meet X^2Y^2 in c^1, then the line joining c^1 to vt is perpendicular to X^2Y^2. In projecting a second plan, all points in it are made the same distances from the new ground line as the corresponding points in the former plan were from the old ground line.

By a continued process of changing the ground line, a projection of

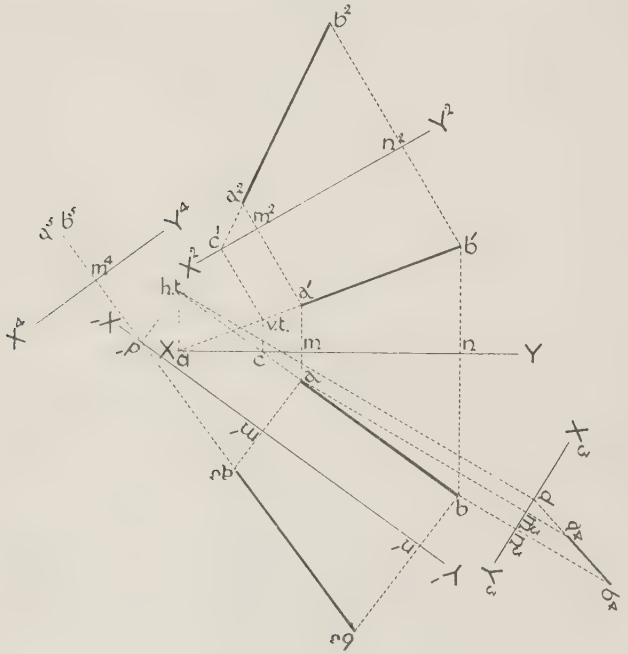


Fig. 99.

the line AB on to a plane making any required angle with it may be obtained as follows: Let an end view of AB be required; then AB must be projected on to a plane to which it is perpendicular. Starting with the given projections ab, a^1b^1, it will be seen that no *one* new projection, either plan or elevation, will give the required result. If, however, a ground line be arranged parallel to either ab or a^1b^1 so as to obtain a full-length projection, as shown at a^3b^3, and a further ground line, making the required angle with a^3b^3 (in this case 90°), is also arranged, as X^4Y^4, a projection on to the latter is the one required. For, the points b^3, a^3, d^1 are all in a straight line perpendicular to X^4Y^4,

and b, a, ht are all equidistant from X^1Y^1; therefore the projections of the extremities of AB are in the same point whose distance from X^4Y^4 is equal to the distance of ab from X^1Y^1. Since a^3b^3 is an elevation, a^5b^5 is a plan, and the trace of the line on the plane X^4Y^4 is in the same point as the projections a^5b^5.

Problem 40. *Fig.* 100.—**To draw the projections of a straight line, given the real length and the inclinations to the planes of projection.**

Let b^1p and b^1q be each equal to the given length, and let θ equal the inclination to HP and ϕ equal the inclination to VP. Let p and

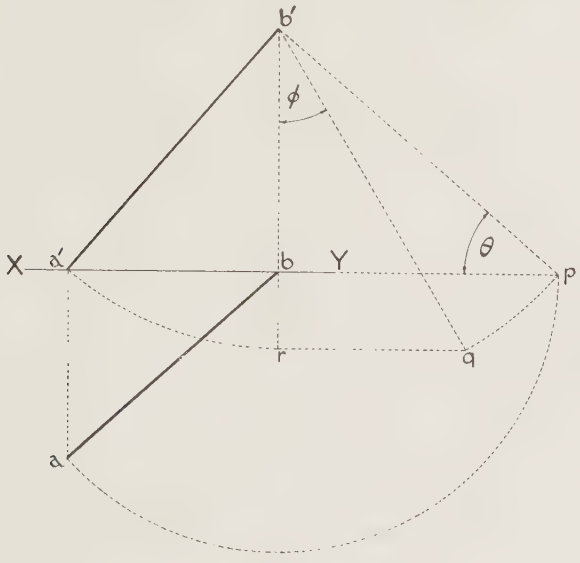


Fig. 100.

XY be in the same straight line, and let b^1r cut the ground line perpendicularly in b. Draw qr parallel to XY. With centre b^1, radius b^1r, cut the ground line in a^1. Draw a^1a perpendicular to XY to cut the circle, centre b, radius bp, in a. Then ab and a^1b^1 are the required projections.

For, if b^1p is the true length and θ the inclination to HP, the required plan must be equal to bp; ba is equal to bp. Also, if b^1q is the true length and ϕ the inclination to VP, b^1r is the length of the required elevation; b^1a^1 is equal to b^1r. And since aa^1 and bb^1 are perpendicular to the ground line, ab and a^1b^1 are the projections of a straight line whose length is equal to bp (and bq) and whose inclinations to HP and VP are θ and ϕ respectively.

The maximum value of θ plus ϕ is 90°. Therefore, when the inclinations are equal b^1p and b^1q will coincide; in all other cases, b^1q will fall below b^1p.

When the projections of any plane rectilinear figure are given, the true shape of the figure may always be determined by finding the true lengths of a sufficient number of sides and diagonals, and then plotting the figure. Generally, however, it is quicker to determine the traces of the plane containing the figure and obtain the true shape by rabatment; this method will be dealt with in a subsequent chapter.

CHAPTER VIII

THE STRAIGHT LINE AND PLANE

(*a*) *A plane may be determined to contain any three given points or any two intersecting straight lines.*

(*b*) *The horizontal trace of any line lying in a plane is in the horizontal trace of the plane, and the vertical trace of the line is in the vertical trace of the plane.*

(*c*) *The plan of any horizontal line is parallel to the horizontal trace of any plane containing the line, and the converse.*

(*d*) *The elevation of any line which lies in a plane and is parallel to the vertical plane is parallel to the vertical trace of the plane, and the converse.*

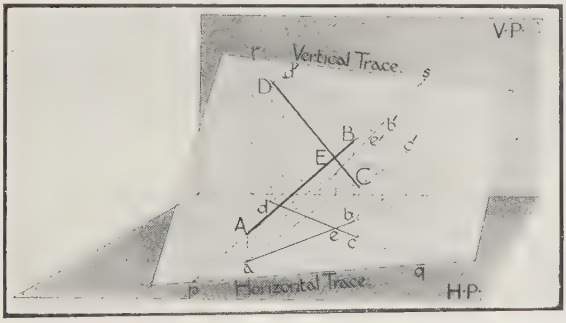


FIG. 101.

Illustrations of (*a*) and (*b*). *Fig.* 101.—Let AB and CD be any two intersecting straight lines; let ab and cd be their plans, and a^1b^1 and c^1d^1 be their elevations. Let p be the horizontal trace of AB, q the horizontal trace of CD, s the vertical trace of AB, and r the vertical trace of CD. Then the plane having p and q in its horizontal trace and r and s in its vertical trace will contain the lines AB and CD. For since the points p, A, B, and s are collinear, and q, D, C, and r are collinear, the plane containing r, s, p, q must contain A, B, C, and D. When three points are given, they may be considered as the angular points of a triangle, and

the traces of any two of the sides will give the traces of the required plane. Consider the three points A, D, and E, and it will be seen that the conditions for three given points are precisely the same as for two intersecting straight lines.

Illustrations of (*c*) and (*d*). *Fig.* 102.—Two straight lines, AB and CD, are shown lying in an oblique plane. The elevation of AB is a^1b^1, and because a^1b^1 is parallel to the ground line, AB is parallel to HP. The plan of CD is cd, and because cd is parallel to the ground line, CD is parallel to VP. Then, because AB is parallel to HP, its plan ab is parallel to the horizontal trace of the oblique plane; and because DC is parallel to VP, its elevation c^1d^1 is parallel to the vertical trace of the oblique plane. For, let the triangle Ppm move so as to have its plane always parallel to VP and m always in ht; then P will be always

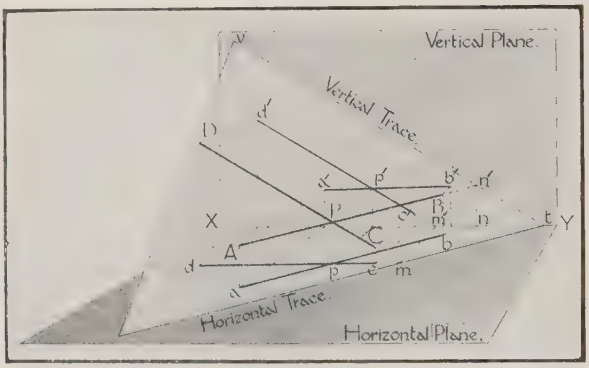


Fig. 102.

in AB and p will move always in ab, and because the length pm and the angle between pm and ht remain constant, ab is parallel to ht; and because the height Pp, the length Pm, and the angle between Pm and ht remain constant, AB is parallel to ht and to the horizontal plane. In the same way, by considering the triangle Pp^1n^1 to move always parallel to HP with n^1 always in vt, P always in CD, and p^1 always in c^1d^1, vt and c^1d^1 may be shown to be parallel and CD be shown to be parallel to VP.

However self-evident the statements (*c*) and (*d*) may appear, the student will do well to give them great attention. Many important practical problems are entirely solved by what is known as the level ordinate method, but the author's experience has shown that it is insufficient merely to see the truth of the statements; it is necessary to seek and create the conditions just explained in most of the problems involving the oblique plane.

Problem 41. *Fig.* 103.—**Given the traces of an oblique plane and one projection of any point contained by the plane, to determine the second projection.**

Let *vth* be the given traces and *p* be the given plan of any point *P* in the plane *vth*.

Draw *pr* parallel to *ht* to meet *XY* in *r*, and from *r* draw a perpendicular to meet *vt* in r^1. Draw a perpendicular from *p* to meet a horizontal from r^1 in p^1; then p^1 is the required elevation. For, since *pr* is parallel to *ht*, *r* and *p* are the plans of points equidistant from *HP*; and since *r* is in *XY*, the elevation of the point whose plan is *r* is in *vt*; but the plan and elevation of a point are in the same straight line perpendicular to the ground line, therefore r^1p^1 is the elevation of the line whose plan is *rp*, and p^1 is the required elevation of *P*.

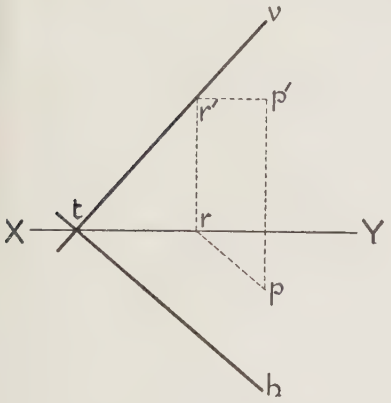


Fig. 103.

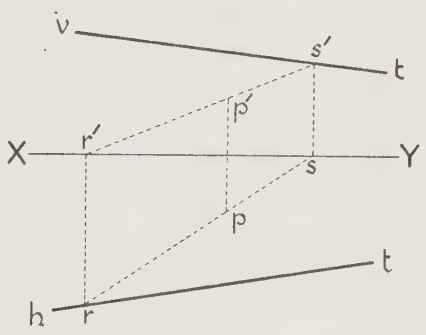


Fig. 104.

Given p^1 the elevation, to find *p*, the plan; draw p^1r^1, r^1r, *rp*, and p^1p in the order given. Or, draw the plan and elevation of the line which is parallel to *VP* and contains *P*; then the perpendicular from p^1 will cut the plan of this line in *p*.

When the traces are only slightly inclined to the ground line the construction shown in Fig. 104 is more applicable, and is as follows:

Through the given projection *p* or p^1 draw any straight line *rs* or r^1s^1, and draw the perpendiculars rr^1 and ss^1; then the perpendicular pp^1 will cut *rs* or r^1s^1 in the required projection. For *rs* and r^1s^1 are the projections of a line lying in the plane and passing through *P*; therefore the projections of the point will be in the projections of the line, and because the plan and elevation of a point are in the same straight line perpendicular to the ground line, if p^1 is the elevation, *p* is the plan, and vice versa.

Parallel lines are equally inclined, their projections are parallel, and the ratio of the lengths of the projections is equal to the ratio of the lengths of the lines.

Illustration. Fig. 105.—Let AB and CD be any two parallel straight lines. Then if Aa, Bb^1, Cc, Dd^1 are each perpendicular to XY, Ba and Ab^1 are the projections of AB, and $DcCd^1$ are the projections of CD. Since Cc is parallel to Aa and DC is parallel to AB, and the angles CcD, Aab are equal—each being a right angle—the triangles DCc, BAa are similar, and $AB : CD = Ba : Dc$. Also the angles CDc, ABa are equal to each other and to the inclination of each line to HP. In a similar manner the triangles Ccd^1, Aab^1 may be

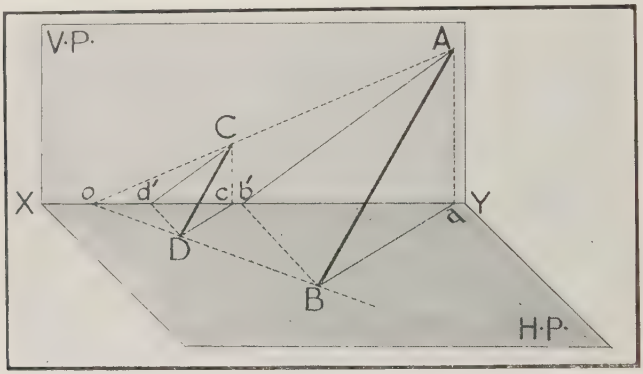


Fig. 105.

shown to be similar, and therefore $AB : CD = Ab^1 : Cd^1$, and the angles DCd^1, BAb^1 are equal to each other and to the inclination of each line to VP.

From the above it will be seen that for any two parallel straight lines, the straight line through the vertical traces will intersect the straight line through the horizontal traces on the ground line; thus, AC produced meets BD produced in O; O being in XY.

If a straight line, lying in one of two intersecting planes and perpendicular to the line of intersection of the planes, be projected on to the second plane, then the projection also is perpendicular to the line of intersection and the angle between the line and its projection is equal to the angle between the planes.

Illustration. Fig. 106.—Let P and P^1 be any two intersecting planes; then, if AB in the plane P is perpendicular to the line of intersection mn, and ab is the projection of AB on to the plane P^1,

ab is perpendicular to *mn* and the angle between *AB* and *ab* is equal to the angle between the two planes *P* and P^1.

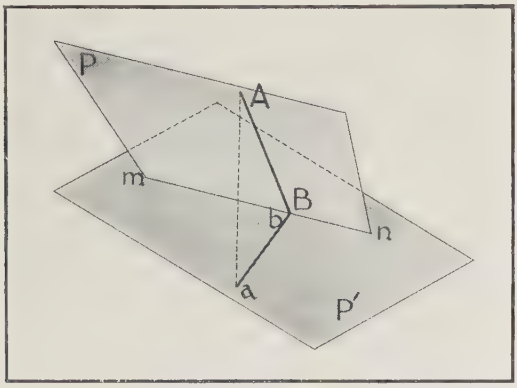


Fig. 106.

Problem 42. *Fig.* 107.—**Given the traces of an oblique plane, to determine its inclinations to the co-ordinate planes.**

Let *vt*, *ht* be the given traces. Draw *mr* perpendicular to *ht*, and from *m* erect a perpendicular to meet *vt* in m^1. With *m* as centre, *mr* as radius, cut *XY* in r^1. Join r^1m^1; then the angle m^1r^1m, marked θ, is the inclination of the oblique plane to *HP*.

To determine the inclination to *VP*; draw sn^1 perpendicular to *vt*, and draw n^1n at right angles to *XY*. With n^1 as centre, n^1s as radius, cut *XY* in s^1. Join s^1n; then the angle ns^1n^1, marked ϕ, is the inclination to *VP*.

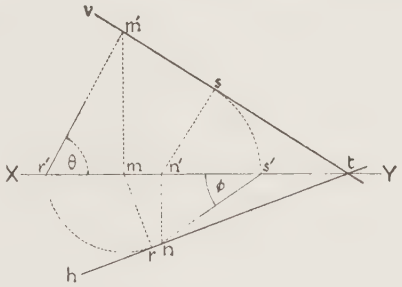


Fig. 107.

For ns^1 is the rabatment of a line which is contained by the oblique plane, is perpendicular to the line of intersection *vt*, and has n^1s as its projection on *VP*. Therefore the projector nn^1, the projection n^1s, and the line ns^1 form the right-angled triangle nn^1s^1. Then ϕ is the angle between the line and its projection on *VP*, as required. In the same way θ may be shown to be the angle between a line, in the oblique plane, and perpendicular to *ht*, and its projection, *mr*, on the horizontal plane.

Problem 43. *Figs.* 108, 109, *and* 110.—**To draw the traces of a plane**

to contain a given line and make a given angle with one of the planes of projection.

Before proceeding to a strictly geometrical solution the pictorial view of the principles involved, Fig. 108, should be carefully studied.

Consider the right circular cone shown in the figure as the surface generated by the right-angled triangle CcM, rotated about the side Cc, cM being always in HP. Then any plane containing the point C and having its horizontal trace tangent to the base of the cone will be inclined to HP at an angle equal to the base angle of the cone, that is, to the base angle of the triangle marked θ in the figure. Let M be the point of contact between the base of the cone and the horizontal

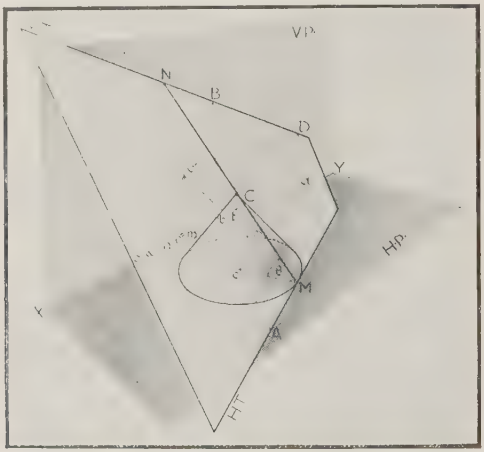


Fig. 108.

trace of a plane containing C; then the generator CM is in the plane, and N, the vertical trace of CM, will be a point in the vertical trace of the plane. Now let A be the horizontal trace and B the vertical trace of any straight line through C; then if the horizontal trace of the plane passes through A, the vertical trace will pass through B. Also, if D is the vertical trace of the horizontal line in the plane through C, D is in the vertical trace of the plane, and cd, the plan of CD, is parallel to HT; the elevation c^1D of CD is parallel to XY, and Dd is perpendicular to XY. The elevation of the cone is ec^1f; Ab and Ba are the projections of AB; and Mn, Nm are the projections of MN. Now consider Fig. 109.

Let ab and a^1b^1 be the projections of any straight line AB, and let it be required to draw the traces of a plane to contain AB and be inclined at θ to HP.

Draw a^1c^1 to make the angle θ with XY. From a draw a line parallel to XY to meet a perpendicular from c^1 in c. With a as centre, ac as radius, draw a circle. Determine q, the horizontal trace of AB, then a tangent from q to the circle, centre a, is the required horizontal trace. Draw ar parallel to ht and erect a perpendicular from r to meet a horizontal through a^1 in r^1; then the required vertical trace passes through r^1 and intersects the horizontal trace in t on the ground line.

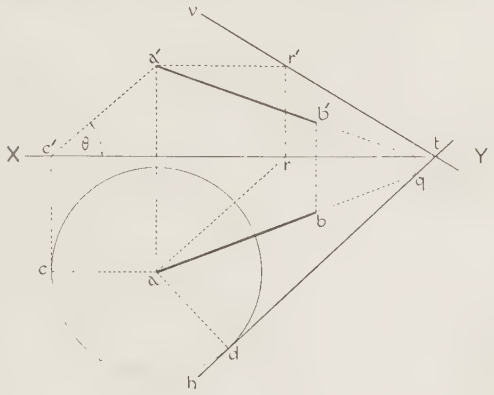


Fig. 109.

Consider the circle, centre a, as the base of a right circular cone having its vertex in A; then a is the plan and a^1 the elevation of the vertex. Since q is the horizontal trace of AB, and ht passes through q and is tangent to the base of the cone, and since (because ar is parallel to ht, a^1r^1 parallel to XY, and r^1 in vt) A is in the plane vth, the plane contains the line AB, the generator Ad, and is inclined to HP at θ, the base angle of the cone.

When both traces of the line are inaccessible, as in Fig. 110, two cones may be introduced, as follows (Fig. 110):

Let ab and a^1b^1 be the projections of a straight line AB, to determine the traces of a plane to contain AB and be inclined to HP at θ.

From a^1 and b^1 draw the straight lines a^1c^1 and b^1d^1 each to make the angle θ with XY. From a and b draw parallels to XY to meet perpendiculars from c^1 and d^1 in points c and d. With centre a, radius ac, and centre b, radius bd, draw the circles to represent the bases of two cones whose base angles are each equal to θ and whose vertices

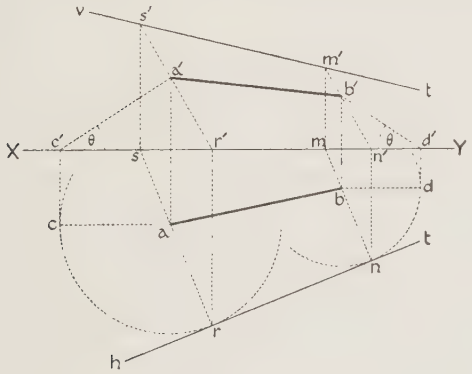


Fig. 110.*

are A and B. Draw ht to touch the base circles in points r and n. Draw a straight line through r and a to meet XY in s, and another through n and b to meet XY in m. Draw perpendiculars to XY from r and n to meet the ground line in r^1 and n^1. Erect perpendiculars from s and m to meet straight lines through r^1, a^1 and n^1, b^1 in points s^1 and m^1; then vt passes through s^1 and m^1. For rs, r^1s^1 and mn, m^1s^1 are the projections of the straight lines which contain the two generators Ar and Bn, and since the extremities of the lines are in the traces of the plane, the plane is tangent to the two cones and is, therefore, inclined at θ, and contains the given line AB.

In Fig. 109 a second tangent to the circle, centre a, may be drawn from q, and would represent the horizontal trace of a second plane fulfilling the required conditions; the vertical trace of this plane would be below the HP, and therefore both traces would occur below the ground line.

In Fig. 110, if the base circles of the cones were completed, a second common tangent could be drawn to represent the horizontal trace of a second plane containing AB and inclined at θ, but since this trace would occur behind VP, in the second dihedral angle, both traces would be above XY. In the Figs. 109 and 110, the more easily understood plane has been selected for the purpose of explaining the principle upon which the solution is based. In practice, however, conditions usually exist which give no opportunity for selection, and the student should make himself capable of determining, and dealing with, either plane. Consider Fig. 111, in which the conditions are similar to those in Fig. 109. ab, a^1b^1 are the projections of the given line AB. p is the horizontal trace and q the vertical trace of AB. a is the centre and ac the radius of the base of the cone whose base angle is θ and whose vertex is in A. Through p draw ht to touch the base circle at r. Join ar and produce to meet XY in s. Draw aa^2 perpendicular to as and equal to ma^1. Join a^2r and produce to meet a perpendicular to as from s in s^1. Make ss^2 perpendicular to XY and equal to ss^1. Draw vt through q and s^2. Then vt and ht are the required traces. For ht contains the horizontal trace of the given line and touches the base circle of

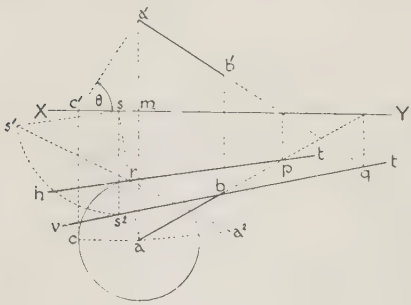


Fig. 111.

the cone whose base angle is θ, in r; hence the required plane must contain the generator, AR, of the cone. Now suppose ra^2 to be a rabatment of the generator AR about its plan, ar; then a^2s^1 is the true length of AR produced to meet VP, and ss^1 is the distance of the vertical trace of AR below XY. But s is the plan of the vertical trace of AR, and because ss^2 is perpendicular to the ground line and equal to ss^1, s^2 is the vertical trace, and since the points s^2, r, p, and q are in the traces of the plane, the plane is tangent to the cone and contains the given line AB.

When two cones envelope a common sphere, the positions of the cones relative to each other being such as to cause the circles of contact between the cones and sphere to intersect, a common tangent plane to the cones and sphere may be found; the plane will touch the sphere in the point

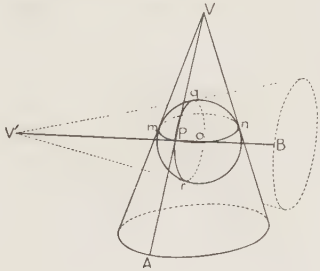


Fig. 112.

of intersection of the contact circles, and touch each cone in the generator which passes through this point.

Illustration. Fig. 112.—Let the two cones, vertices V and V^1, touch the sphere, centre o, in circles mPn and qPr, and let P be the point of intersection of these circles. Let VA and V^1B be the generators —one in each cone—that pass through P. Then the plane containing these two generators is tangent to each cone and to the sphere. For the lines VA and VB are tangents to the sphere at P, and since P is the point of intersection of the contact circles, P is a point in the surfaces of both cones, and since, also, no plane through V and V^1 may cut the cones without cutting the sphere, the plane containing VA and VB touches the cones in these generators and touches the sphere at P.

Consider Fig. 113, in which a tangent plane to two cones and an inscribed sphere is shown in orthographic projection. The projections of the vertices of the cones are shown at a, a^1 and b, b^1. The base of the cone, vertex A, has its base, centre a, radius ac, in HP, and the cone, vertex B, has its base, centre b^1, radius b^1d^1, in VP. The pro-

jections of the centre of the sphere are at a and b^1. The horizontal trace of the line containing A and B is at g, the vertical trace is at e. The points e and g are determined by making aA and bB equal to fa^1 and fb^1 respectively, and perpendicular to ab, and extending AB to

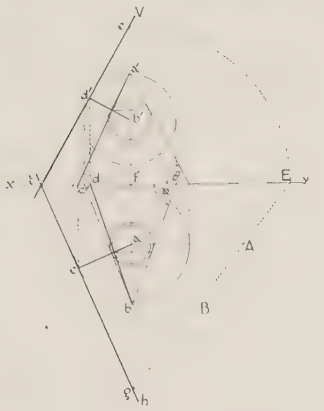


Fig. 113.

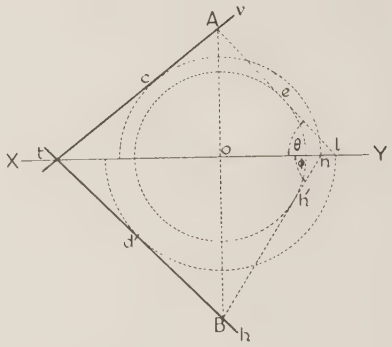


Fig. 114.

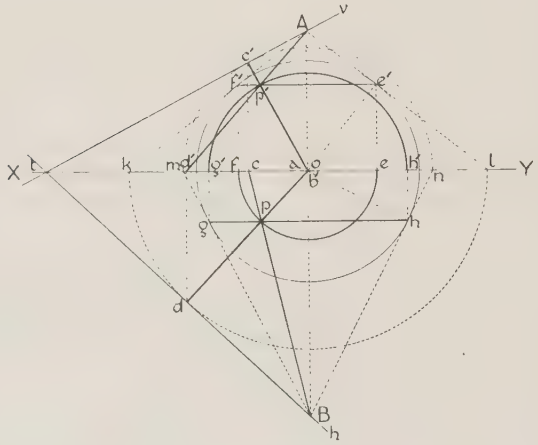


Fig. 115.

meet the ground line in E and ab produced in g, and making fe equal to fE ; e being in a^1b^1 produced. The traces of the required plane pass through g and e and are tangent to the base circles at c and d^1. Then the plane vth touches one cone along the generator AC and the other along the generator BD, and the inclinations of the plane are equal to the base angles of the cones, *i.e.*, θ to HP and ϕ to VP.

Problem 44. *Figs.* 114 *and* 115.—**To draw the traces of a plane inclined at θ to *HP* and ϕ to *VP*.**

Consider Fig. 114. With any point o in XY as centre, and with any convenient radius, draw a circle, eh. Draw tangents nh^1 and le to this circle, making the given angles with XY; θ above and ϕ below, as shown. Through o perpendicular to XY draw a straight line to meet the tangents in A and B. With centre o, radii on and ol, draw the semicircles ld and nc. Then tangents to these semicircles from A and B will intersect on XY and are the required traces.

Consider Fig. 115, in which the circle, centre o, containing the points g, h, e^1, f^1, is the plan and elevation of a quarter sphere, having its centre in the ground line and its semicircular faces one in each plane of projection. Akl is the elevation of a half cone, standing with its semicircular base kdl in HP, and its axis in VP, touching the quarter sphere in the semicircle whose plan is fpe and whose elevation is $f^1p^1e^1$. Bmn is the plan of a half cone having its semicircular base mc^1n in VP, and its axis in HP, touching the quarter sphere in the semicircle whose plan is gph and whose elevation is $g^1p^1h^1$. Tangents from A and B touch the base semicircles in c^1 and d, and intersect in t on XY. Let these tangents be the traces of an oblique plane; then, because they touch the base semicircles at d and c^1 and contain the vertices of the half cones, the plane contains the generators—one in each cone—whose projections are da, d^1A and Bc, b^1c^1; and because the half cones touch the quarter sphere in the contact semicircles FE and GH and these semicircles intersect in P, the plane touches the two half cones in lines of contact through P. Compare with Fig. 114.

CHAPTER IX

RABATMENT OF PLANES; PERPENDICULARS TO INCLINED AND OBLIQUE PLANES; PROBLEMS ON PIPE BENDING

To obtain the true shape of a figure lying in a plane, the plane may be rabatted about either its vertical trace or its horizontal trace; in the case of an oblique plane, a double rabatment may be made, first about a new ground line to obtain an edge view of the plane, and then about the edge view. The special circumstances of any particular case must decide which of the methods is the most suitable. The general principles on which the methods of rabatment are based are illustrated in Figs. 116 and 117.

In Fig. 116 *vth* are the traces of an oblique plane, and it is required to obtain an edge view of the plane. This operation is sometimes referred to as "converting an oblique plane into an inclined plane." X^1Y^1 is any straight line drawn perpendicular to the horizontal trace of the given plane. The perpendicular aa^1 is drawn from any point a in XY to meet vt in a^1. From a the line ab^1 is drawn parallel to ht to cut X^1Y^1 in b. The distance bb^1 is made equal to aa^1, and b^1 is joined to t^1, the point of intersection between ht and X^1Y^1. v^1t^1 is an edge view of the plane, and the inclination of v^1t^1 to X^1Y^1 is equal to the inclination of the plane to HP. For since ab is the plan of a horizontal line in the plane—being parallel to ht—and aa^1 is the height of this line above HP, t^1b^1 is the rabatment of the line about its plan, and is the line of intersection between the plane *vth* and a vertical plane perpendicular to it.

Problem 45. *Fig.* 117.—**To rabat any oblique plane about one of its traces. Let *vth* be the traces of any given oblique plane.**

From any point a^1 in vt drop a perpendicular to meet XY in a. From a, perpendicular to ht, draw a straight line to meet a circle, centre t, radius ta^1, in A. Then tv^1, through A, is the rabatment of vt about ht, and A is the rabatment of a^1, and α is the real angle between the traces of the given plane.

NOTE.—If aA^1 is perpendicular to ab and equal to aa^1, A^1b is the true length of the line in the plane whose plan is ab, and bA is a rabatment of this line; therefore bA equals bA^1. This gives a second method of solution.

To rabat the plane about vt, draw aa^1 with a^1 in the ht, and ab perpendicular to vt. The working will be seen clearly if the figure is reversed so as to bring ht above and vt below the ground line. There is no fundamental difference between the relationships of the co-ordinate planes to any oblique plane; and the expedient of viewing a problem in the reverse position may frequently assist the inexperienced student.

The following theorem relates to the movement of point A in the

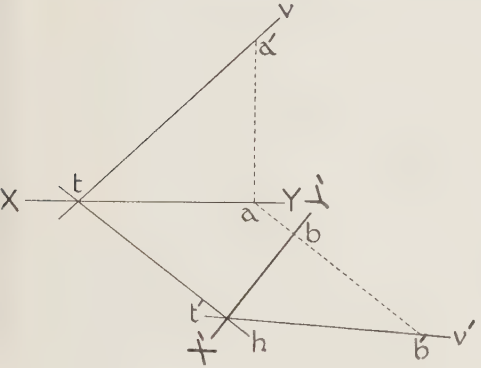


FIG. 116.

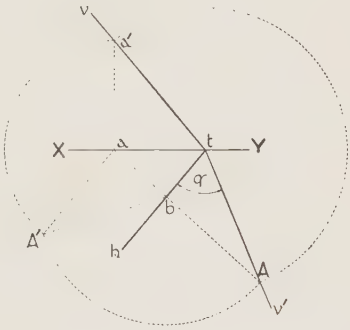


FIG. 117.

foregoing problem and is of great importance in many of the subsequent practical problems.

If any point or any system of points be turned about any straight line as axis, the positions of the points relative to each other and to the axis being constant, the projection of the path of each point on to any plane containing, or parallel to, the axis of rotation is perpendicular to the axis.

Illustration (1). *Fig.* 118.—Let the triangle ABC rotate about the fixed line MN; then each of the angular points A, B, C will describe a circle whose centre is in MN and whose plane is perpendicular to MN, and the projection of each circle on to the plane containing MN is a straight line perpendicular to MN. Thus, if MN is in the plane of the paper, the projection of the circular path of the point A is the straight line AA^2; the projection of the path of B is BB^2, and the path of C projects into the line CC^2. The projections of the semicircles described by the angular points when the triangle turns through 180°,

on a plane perpendicular to MN, are shown at aa^1a^2, bb^1b^2, and cc^1c^2, and the three positions of the triangle shown in plan at ABC, $A^1B^1C^1$, $A^2B^2C^2$ are shown in elevation at abc, $a^1b^1c^1$, and $a^2b^2c^2$.

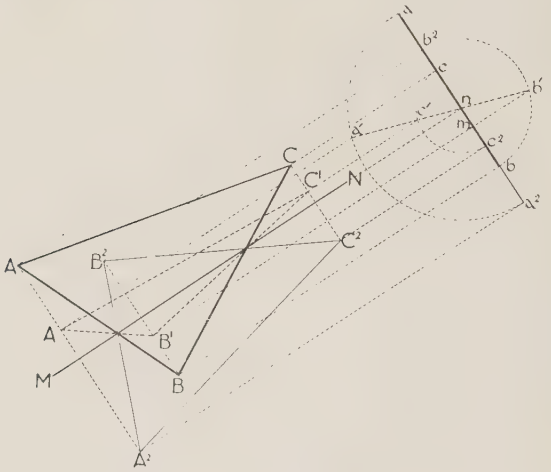


Fig. 118.

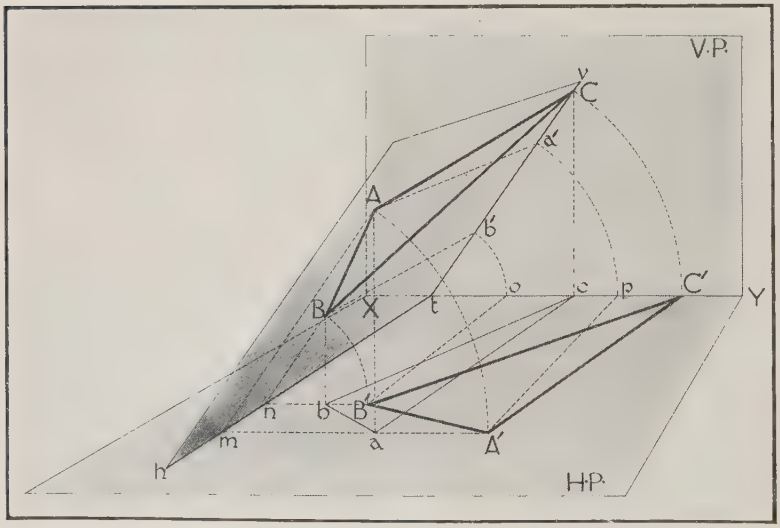


Fig. 119

Illustration (2). *Fig.* 119.—Let ABC be a triangle lying in the inclined plane *vth*, *ht* being at right angles to XY. Then perpendiculars to HP from A, B, and C meet HP in points a, b, and c, and the triangle

abc is the plan of *ABC*. Since *vth* is perpendicular to *VP*, the elevation of the triangle is in *vt*; a^1 being the elevation of *A*, b^1 the elevation of *B*, and *C* is actually in *vt*. Aa^1, Bb^1, and *ht* are parallels, each being a perpendicular to *VP*. Suppose the plane *vth* to turn about *ht* until it coincides with *HP*; then *A* will describe the circular arc AA^1, centre *m*, *B* the circular arc BB^1, centre *n*, and *C* the circular arc CC^1, centre *t*; the lines A^1m, B^1n, C^1t are perpendiculars to *ht* and pass through the points *a*, *b*, and *c* respectively. The arc CC^1 is in *VP*, and the projections of the arcs AA^1, BB^1 on to *VP* are a^1p and b^1o, and since $ta^1 = mA$, $tb^1 = nB$, A^1p and B^1o are parallel to *ht*. $A^1B^1C^1$ is the triangle

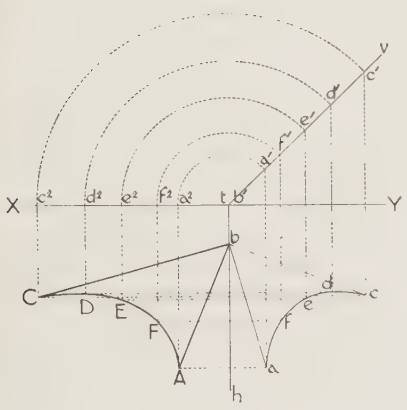


Fig. 120.

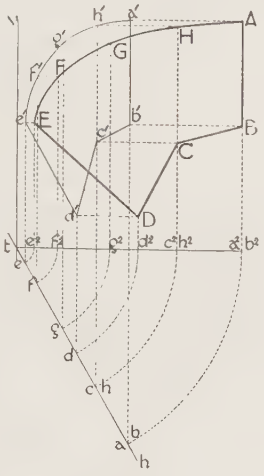


Fig. 121.

ABC rabatted into *HP*, and would be the true shape of the triangle in orthographic projection.

Problem 46. *Figs.* 120 *and* 121.—**Given the projections of a figure lying in an inclined plane, to determine the true shape of the figure.**

In Fig. 120 *abc* is the plan of a figure lying in the plane *vth*. Select any points *d*, *e*, *f* in the curved portion of the outline and project all the points on to *vt* in a^1, b^1 . . . f^1. With *t* as centre turn these points into *XY* at a^2c^2 . . . f^2. From a^2 draw a perpendicular to *XY* to meet a perpendicular to *ht* from *a* in *A*. Repeat this operation for each point to obtain *ACD* . . . *F*. Then *AbCDEF* is the required true shape.

In Fig. 121 the plane containing the given figure is perpendicular to *HP* and inclined to *VP*. The diagram should require no further explanation.

Problem 47. *Fig.* 122.—**Given the projections of the lines of intersection between a pitched roof and the roofs over two dormer windows, one of the latter being semicircular, the other comprising two plane surfaces intersecting in a horizontal ridge, to develop the lines of intersection.**

On the left of the figure dch, $d^1c^1h^1$ are the projections of the lines of intersection between the roof and the ridged dormer gh, g^1h^1 being the projections of the ridge. With h^1 as centre, h^1c^1 as radius, cut h^1g^1 produced in c^2; drop a perpendicular from c^2 to meet perpendiculars to

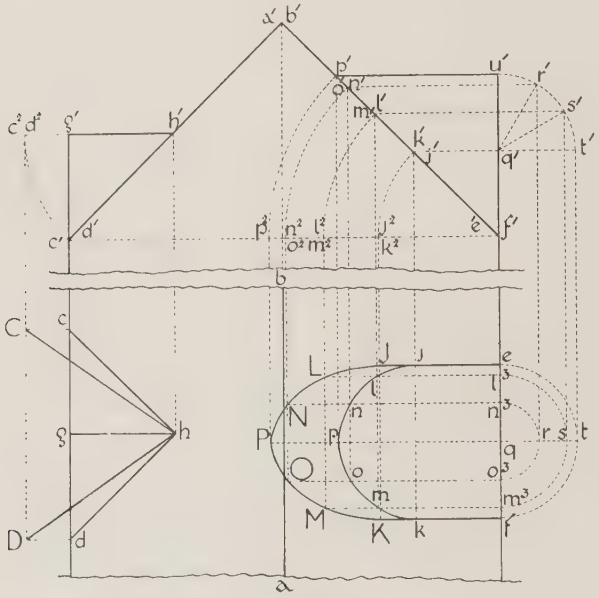


Fig. 122.

cd from c and d in C and D. Then Ch and Dh are the real lengths of the lines of intersection, and ChD is the real angle between them.

On the right of the figure, f^1p^1 is the elevation of the line of intersection, b^1q^1 being the elevation of the springing line of the semi-circular dormer. Let qp be the centre line in plan; with q^1 as centre, q^1u^1 as radius, draw the quarter circle u^1t^1 and on it select any number of points, as r^1 and s^1. Produce pq to intersect perpendiculars from r^1, s^1, t^1 in r, s, t. With q as centre turn r, s, and t to e, f, l^3, n^3, o^3, m^3. From r^1, s^1 draw horizontals to meet f^1p^1 in o^1n^1, m^1l^1. From l^3, n^3, o^3, m^3 draw parallels to pq to meet perpendiculars from n^1o^1, m^1l^1 in points l, m, n, o in the plan of the line of intersection. With f^1 as

centre turn the points in f^1p^1 to p^2, o^2, n^2, l^2, m^2, j^2, k^2. Then perpendiculars from the points in f^1p^2 will meet parallels to pq from j, k, l, m, n, o, p in J, K, L, M, N, O, P; the straight lines eJ, fK and the semi-ellipse KPJ give the true shape of the line of intersection.

Problem 48. *Fig.* 123.—**To determine the real angle between any two straight lines whose projections are given.**

Let ab, a^1b^1 and cd, c^1d^1 be the given projections. then p, p^1 are the projections of the point of intersection. Determine T and T^1, the horizontal traces of the given lines; then the straight line through these points is the horizontal trace of the plane containing the lines. Draw X^1Y^1 at right angles to TT^1 (not necessarily through T) and from p draw pp^2, perpendicular to X^1Y^1 and intersecting it in r^1, making r^1p^2 equal to rp^1. Tp^2 is an edge view of the plane containing the given lines. With T, the point of intersection of the traces, as centre, Tp^2 as radius, cut X^1Y^1 in q. From q, parallel to TT^1, and from p, perpendicular to TT^1, draw straight lines to intersect in P. Join PT and PT^1, then the angle TPT^1 is the real angle between the lines. The true lengths of the lines may be determined by drawing cC, dD, aA, and bB perpendicular to TT^1.

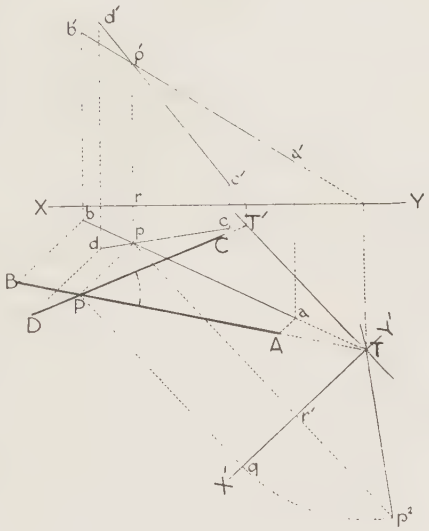


Fig. 123.

The following problems in this and succeeding chapters that are considered in a practical form involve geometrical principles that have a very general application; the student should not ignore them because they appear to deal with a section of building apart from that in which he is specialising. The only difference between an ordinary geometric problem and a practical application of geometry is in the terms used, and any practical problem may be stated in strict geometrical terms without altering its character in the least degree. For many students a problem set in practical terms has the advantage of helping towards the necessary clear visualisation of the conditions.

Problem 49. *Figs.* 124 *and* 125.—**Given the plan and elevation of an inclined metal pipe bent around the angle between two vertical wall surfaces, to determine the real angle between the two straight sections of the pipe.**

Fig. 124.—Let *abc* and $a^1b^1c^1$ be the projections of the centre lines of the two branches of the pipe, the wall surfaces in this example being at right angles. Draw cc^3 perpendicular to *bc* and equal to t^1c^1 (the difference in the heights of a^1 and c^1). Make bb^3 equal to t^1b^1 (the difference in the heights of a^1 and b^1). Through c^3 and b^3 draw a straight line to meet *cb* produced in *t*; then the straight line through *a* and *t* is the

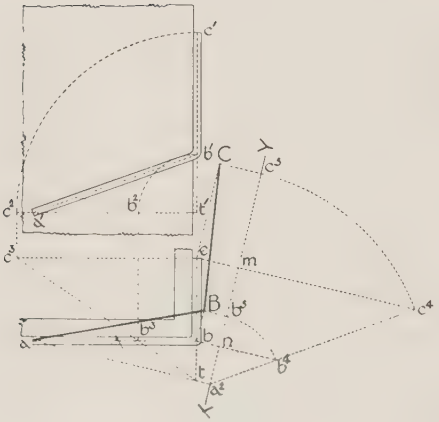


Fig. 124.

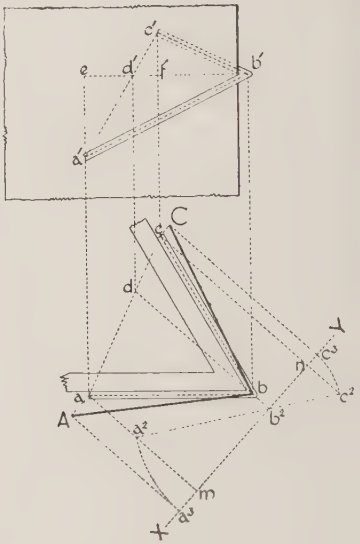


Fig. 125.

horizontal trace of the plane containing the centre lines. Draw *xy* perpendicular to *at*, and draw cc^4 and bb^4 perpendicular to *xy* and intersecting it in *m* and *n*, making mc^4 equal to t^1c^1 and nb^4 equal to t^1b^1. A straight line through c^4 and b^4 will intersect *at* in a^2 on *XY*, and a^2c^4 is the vertical trace of the plane containing the centre lines. With a^2 as centre, a^2b^4 and a^2c^4 as radii, cut *XY* in b^5 and c^5. Parallels to *XY* from *b* and *c* intersect perpendiculars to *XY* from b^5 and c^5 in points *B* and *C*. The required angle is *aBC*.

Fig. 125.—A second method is shown and in this example the vertical wall surfaces make an acute angle. The projections of the centre lines of the two branches of the pipe are *abc* and $a^1b^1c^1$. Join *a*, *c* and a^1, c^1. From b^1 draw a horizontal line to cut a^1c^1 in d^1 and cut

perpendiculars through a, a^1 and c, c^1 in e and f. Drop a perpendicular from d^1 to meet ac in d; then bd and b^1d^1 are the projections of a horizontal line through the point of intersection of the centre lines. Draw XY perpendicular to db, and from a and c draw perpendiculars to XY and meeting it in m and n. Make ma^2 equal to ea^1 and nc^2 equal to fc^1. Draw a^2c^2 to intersect db produced in b^2 on XY; then a^2c^2 is an edge view of the plane containing the centre lines. With b^2 as centre, b^2a^2 and b^2c^2 as radii, cut XY in a^3 and c^3. From a and c draw parallels to XY to meet perpendiculars to XY from a^3 and c^3 in A and C. Then the required angle is AbC.

In the foregoing problems on pipe bending it was assumed that the directions shown were necessary to avoid a window opening, or for some similar reason. In a great many cases, however, in which a pipe or wire casing is required to be run along two or more flat surfaces, the condition is simply that of connecting two fixed points; it is then necessary to determine the directions that give the shortest distance between the points. If the two points were in the same plane, the shortest distance would of course be a straight line, and it is *by creating this simple condition* that the directions are determined for points in different surfaces. The following problem will show how this may be done.

Problem 50. *Fig.* 126.—**Given the projections of any two points, lying one in each of two intersecting planes, to determine the shortest distance, measured along the surfaces of the planes, between the points.**

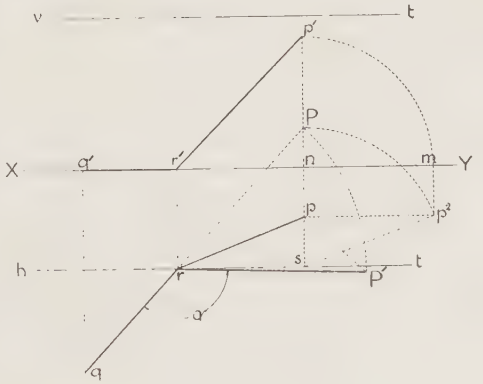


Fig. 126.*

Let p, p^1 be the projections of a point P in the plane vt, ht; and let q, q^1 be the projections of a point Q in the horizontal plane.

Through p^1, p draw a straight line to meet ht in s and cut xy in n. With n as centre turn p^1 to m, and from p parallel to XY draw pp^2 equal to nm; then pp^2 equals np^1, equals the height of P above HP. With s as centre turn p^2 to P. Join P and q; then r, where Pq cuts ht, is the intersection of two straight lines qr, pr, one in each plane, such that the sum of the distances qr, pr is the shortest distance between

P and Q, measured along the surfaces of the planes. For let r be in any other position in ht; then since P is the rabatment of the given point about ht, the two distances Pr and qr would not lie in the same straight line, and their sum would therefore be greater than the straight line qP.

Problem 51. *Fig.* 126.—**Having determined the least distance between the two points P and Q, as in the foregoing problem, to determine the real angle between the lines Pr and Qr.**

From p, perpendicular to qr, draw pP^1, making rP^1 equal to rP. Join P^1r; then $qrP^1(\alpha)$ is the required angle. For qr is in HP, and qP is the sum of the distances Qr, Pr. Then if rP is rotated about qr into HP, the locus of P will be perpendicular to qr, and since qr remains in position in HP, when rP^1 is equal to rP, P^1 being a point in the locus, both lines are in HP and the true angle is seen.

Problem 52. *Fig.* 127.—**Given the positions of two points in a room, one on the horizontal floor surface, the other on the vertical wall surface, to determine the shortest length of pipe that will lie in the surfaces and connect the given points, and to determine the position and magnitude of the bend between the two sections of the pipe.**

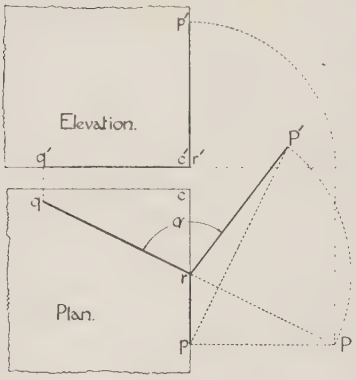


Fig. 127.

Let q, q^1 be the projections of the point on the floor, and p, p^1 the projections of the point on the wall. Draw pP perpendicular to cp and equal to c^1p^1. (c^1p^1 is the height of the point on the wall above the floor.) Draw the straight line qP to intersect the plan of the line of intersection between the wall and floor in point r; then r is the position of the bend and qP the total length of pipe required. From p, perpendicular to qP, draw a straight line to intersect a circular arc, centre r, radius rP, in P^1; then the angle qrP^1 (α) is the required bend in the pipe at r.

In the foregoing examples on pipe work, the points are arranged on adjacent planes or surfaces so as to involve one bend only; but in practice cases frequently arise in which two or more bends are necessary and the pipe is required to be run over several surfaces. This condition introduces an additional problem; for, besides finding the angles between the straight sections, it is necessary to determine the directions of certain sections relative to the plane containing two other sections.

PLATE I.

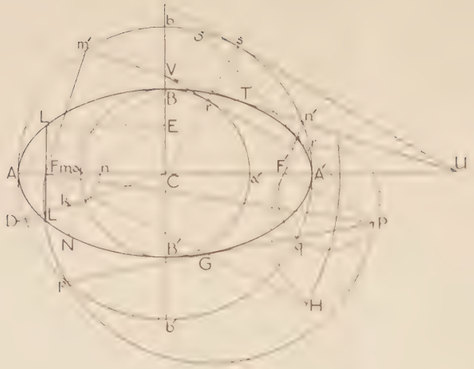


FIG. 63.*

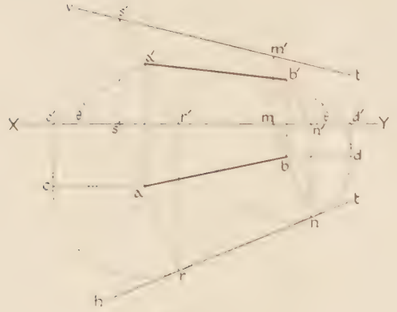


FIG. 110.*

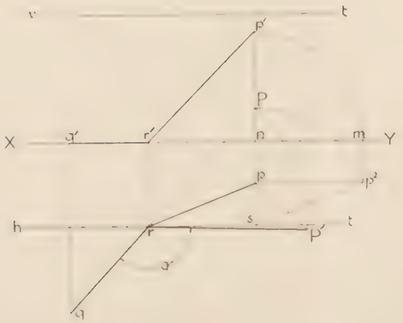


FIG. 126 *

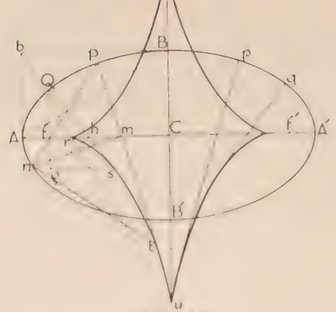


FIG. 68.*

FIG. 69.*

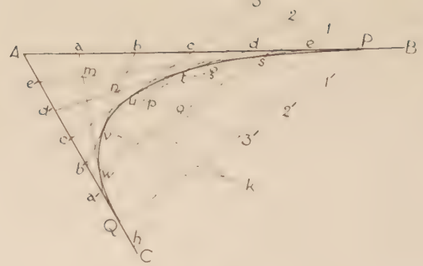


FIG. 78.*

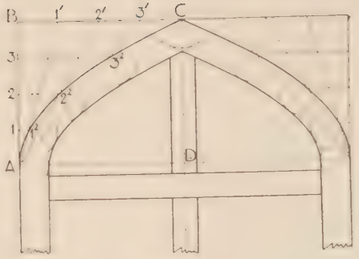


FIG. 79.*

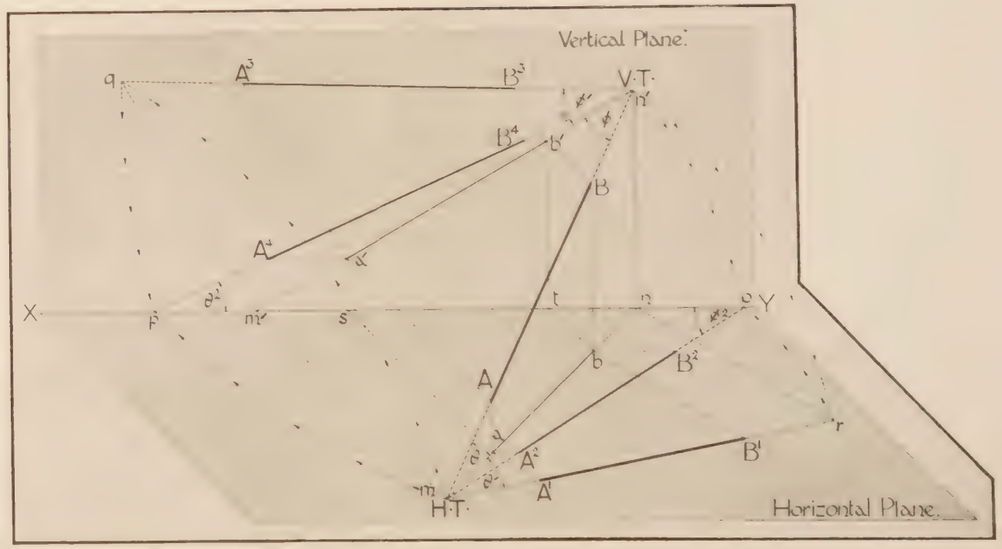


FIG. 94.*

Face page 104.

Before introducing this additional problem it is necessary to draw attention to the following theorems:

If two straight lines containing a right angle are projected on to any plane parallel to one of the lines, the angle between the projections is also a right angle.

Illustration. *Fig.* 128.—Let the right-angled triangle *ABC* turn about the side *AC*, *AC* being in or parallel to the plane of the paper; then, since BB^1, the projection of the locus of *B*, is perpendicular to *AC*, the projection of *BC* is always at right angles to *AC* or its projection on the paper. The rabatment of the circular locus and three

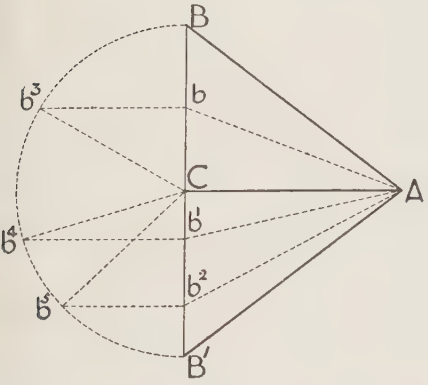


Fig. 128.

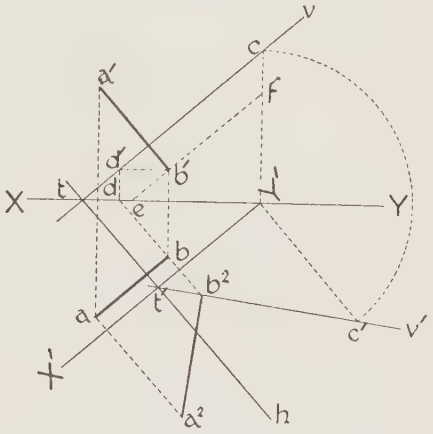


Fig. 129.

positions of the triangle while moving through 180° are shown in the figure.

Any straight line which is perpendicular to a plane has its projections perpendicular to the traces of that plane.

Illustration. *Fig.* 129.—Let *ab*, a^1b^1 be the projections of a straight line perpendicular to the plane *vth*, and let *b* and b^1 be the projections of the point of intersection between the line and plane. Then every line in the plane through *B* will be at right angles to *AB*. Let *DB* be a horizontal line in the plane, then its plan *db* will be parallel to *ht*; and because *ABD* is a right angle and *DB* is parallel to *HP*, the angle between the projections *ab*, *db* will be a right angle; and since *ht* and *db* are parallel, *ab* is perpendicular to *ht*. Now let *ef* be the elevation of a line in *vth*, parallel to *VP* and passing through b^1. Then *AB* is perpendicular to *EF*, and because *EF* is parallel to *VP*, a^1b^1f is a right angle, and since b^1f is parallel to *vt*, a^1b^1 is perpendicular to *vt*.

An edge view of the plane and line is shown at t^1v^1 and a^2b^2, and it should be noted that the point of intersection between the line and plane is determined from this view.

Problem 53. *Figs.* 130 *and* 131.—**To determine the real angle between a given plane and a given straight line.**

Let *vth* be the traces of the given plane and *ab*, a^1b^1 the projections of the given line.

It has already been explained that the angle between a line and a plane is the angle between the line and its projection on the plane. In Fig. 130 *vth* is an inclined plane and one extremity of the given line is in the plane at *a*, a^1. Draw b^1b^3 perpendicular to *vt*, and from *b*, perpendicular to *ht*, draw a line to meet a perpendicular to *XY* from b^3

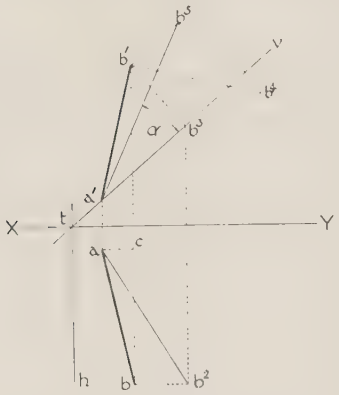


Fig. 130.

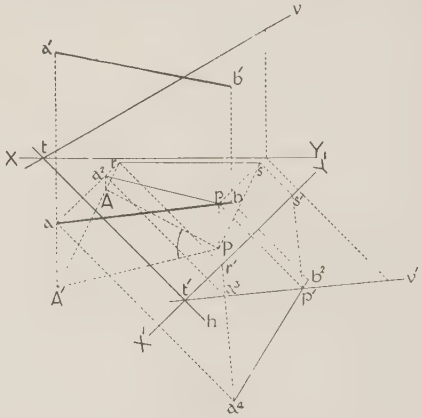


Fig. 131.

in b^2. Join *a*, b^2; then ab^2, a^1b^3 are the projections of *AB* on *vth*. (bb^2 and b^1b^3 are the projections of a perpendicular to the plane from *B*; b^2, b^3 being the projections of the point in which it meets the plane. Compare with Fig. 129.) The problem is now reduced to that of finding the angle between two straight lines whose projections are *ab*, a^1b^1 and ab^2, a^1b^3; but the problem is simplified by regarding the two lines as two sides of a right-angled triangle, the third side being the perpendicular to the plane from *B*. Because the perpendicular is parallel to *VP*, a^1b^3 is its true length and the angle at b^2b^3 is a right angle. Through *a* draw *ac* parallel to *XY* and draw b^1b^4 perpendicular to a^1b^1 and equal to *cb*. With a^1 as centre, a^1b^4 as radius, draw an arc to meet a parallel to *vt* in b^5. Join a^1b^5; then α is the angle between the line and plane.

Note.—A perpendicular to *vt* from b^5 would complete the right-angled triangle which has *AB* as hypotenuse, the projection of *AB*

on *vth*, and the perpendicular on to the plane from *B* as its right-angled sides.

In Fig. 131 the traces of an oblique plane *vth*, and the projections of a straight line *ab*, a^1b^1 are given; to determine the inclination of the line to the plane. Draw X^1Y^1 perpendicular to *ht* and on this as ground line determine an edge view of the given plane as v^1t^1, and a second elevation of the given line, a^4b^2. Then p^1, where a^4b^2 cuts v^1t^1, is an elevation of the point of intersection between the line and plane, and a perpendicular to X^1Y^1 from p^1 meets *ab* in *p*, the plan of the point of intersection between the line and plane, and a perpendicular to X^1Y^1 from p^1 meets *ab* in *p*, the plan of the point of intersection. From a^4 draw a perpendicular to v^1t^1 to meet X^1Y^1 in r^1 and cut v^1t^1 in a^3. From p^1 draw a perpendicular to v^1t^1 to meet X^1Y^1 in *s*. From *a* and *p* draw perpendiculars to *ht* to meet perpendiculars to X^1Y^1 from a^3, r^1, and s^1, in points a^2, *r*, and *s*; then a^2p is the projection of the given line on the given plane and *rs* is the horizontal trace of the plane containing the given line and its projection. Draw aA^1, a^2A, and *pP* perpendicular to *rs*, and make rA^1 equal to r^1a^4, *rA* equal to r^1a^3, and *sP* equal to s^1p^1. Then APA^1 is the required angle between the given line and the given plane. For *ar* and *ps* are the plans of two lines perpendicular to the given plane, meeting *HP* in *r* and *s* and intersecting the given line in *A* and *P*; and since the vertical plane on X^1Y^1 is parallel to these lines, the projections a^4r^1 and p^1s^1 are equal to the real lengths of these lines, and a^3a^4 is the actual distance of *A* from the plane. *P* is in the plane. Let the given line and its projection be turned into *HP* about *rs* as axis; then the plans of the loci of the points whose projections are *a*, a^2, *p* will be perpendicular to *rs*, as at *pP*, aA^1, a^2A, and *sP* will equal s^1p^1, *rA* will equal r^1a^3, and rA^1 will equal r^1a^4, and the triangle APA^1 is the true shape of the triangle whose plan is apa^2. Therefore the angle APA^1 is the angle between the given line and its projection on to the given plane.

Problem 54. *Figs.* 132 *and* 133.—**Given two points in a room, one on the floor, the other in the surface of the horizontal ceiling, to determine the directions of the shortest length of pipe to lie in the floor, vertical wall, and ceiling surfaces, and connect the two points. To determine, also, the positions and magnitude of the two bends and the angle between one section of the pipe and the plane containing the two other sections.**

First consider Fig. 133, in which a length of pipe *QP*, bent at the positions *R* and *S*, is shown with the two sections *SR* and *RP* lying on the surface of a bench, while the section *SQ* inclines upwards so as to make the angle γ with the bench surface.

Let α and β be the angles between the sections at R and S; then it will be seen that γ may be made to vary without changing the magnitudes of either α or β, since SQ may be in any position in the surface of a right circular cone having its axis in the same straight line with SR, and its vertex angle equal to $360° - 2\beta$. It is necessary therefore to determine in practice the relative directions of the sections in addition to the magnitudes of the bends. Refer now to Fig. 132.

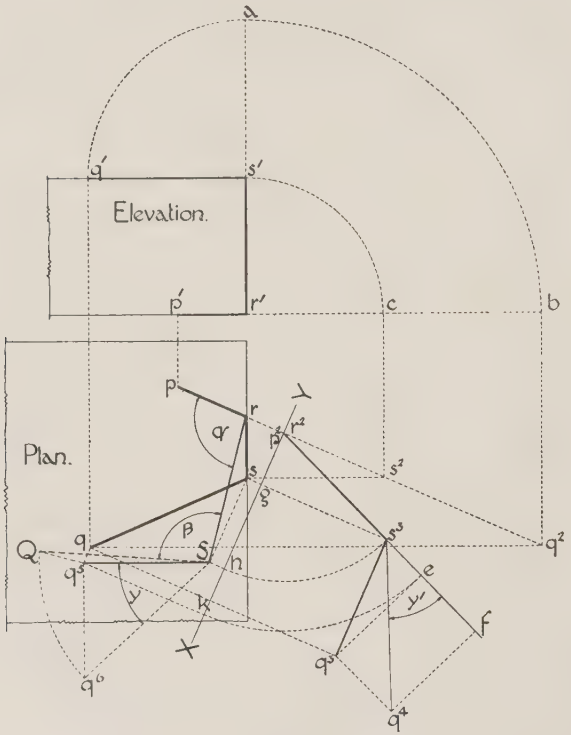


Fig. 132.

Let p, p^1 be the plan and elevation of a point P on the floor, and q, q^1 the plan and elevation of a point Q on the ceiling of a room. It is required to run a pipe from P to Q along the floor, wall, and ceiling surfaces, using the minimum length of pipe. Since the pipe must lie in floor, wall, and ceiling surfaces, its elevation will be $p^1r^1s^1q^1$. Produce p^1r^1 to c and b, making r^1c equal to r^1s^1, and cb equal to s^1q^1. Draw qq^2 parallel to p^1b, making bq^2 perpendicular to p^1b. Join p, q^2. Then pq^2 is the length of pipe required. Draw a perpendicular to p^1b from c to meet pq^2 in s^2; then r and s^2 are the positions of the required bends.

From s^2, parallel to p^1b, draw a straight line to meet the plan of the vertical wall surface in s; then $prsq$ is the plan of the pipe when in position. Because pr is in the horizontal floor surface, pq^2 is the horizontal trace of the plane containing the two sections whose plans are pr and rs. Draw XY in any convenient position perpendicular to pq^2, and draw ss^3 parallel to pq^2 to cut XY in g, and make gs^3 equal to r^1s^1. From r^2, where XY cuts pq^2, draw a straight line through s^3; then r^2s^3 is the vertical trace of the plane containing the two sections whose plans are pr and rs. From q, perpendicular to XY, and from s^3, parallel to XY, draw straight lines to intersect in q^3, then $p^2r^2s^3q^3$ is an elevation of the bent pipe in a direction at right angles to the plane containing the two sections PR and RS. Project q^3 on to the plane at e. With

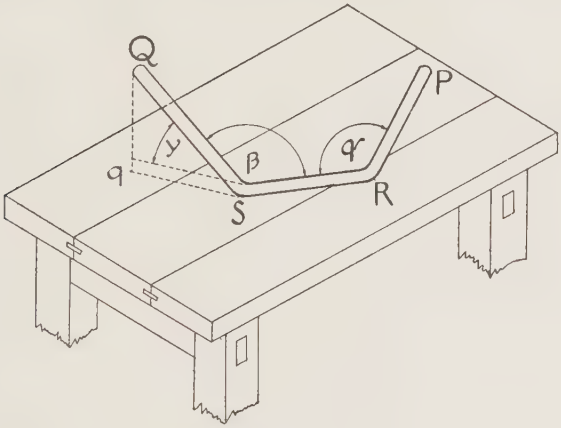


Fig. 133.

p^2 as centre, turn s^3 and e to h and k. From h and k, perpendicular to XY, and from s and q, parallel to XY, draw straight lines to intersect in S and q^5; then $prSq^5$ is the plan of the bent pipe on the bench surface, when the sections PR and RS are lying on that surface and Q is above the surface an amount equal to eq^3. In Fig. 133 the pipe is shown in the latter position, $PRSq$ being the plan on the surface of the bench and qQ the height of Q above the surface. To determine the bend at S, Fig. 132, make SQ equal to sq, with q^5Q perpendicular to rS. Or, make q^5q^6, perpendicular to Sq^5, equal to eq^3, and turn q^6 about S to Q. The angle q^5Sq^6, marked γ, is the inclination of QS to the bench surface, as shown in Fig. 133. The angle γ may be determined by a second method, as follows: Draw q^3q^4 parallel to r^2e and make s^3q^4 equal to qs. The suggested procedure for making the bends is as follows: Set out on the bench or floor surface the lines $prSq^5$. At a distance from one end

equal to pr make the bend α. Fix the pipe on the bench so as to make the bend conform to the plan, and make the bend at S by raising the end of the pipe to stand directly over q^5 an amount equal to eq^3. The actual bend at S should now equal β.

Problem 55. *Fig.* 134.—**Let it be required to run a pipe from P on the ground to U on the further vertical wall surface, the pipe to lie in wall and roof surfaces. To determine the minimum length, the positions and magnitudes of the bends, and the distances of the angles**

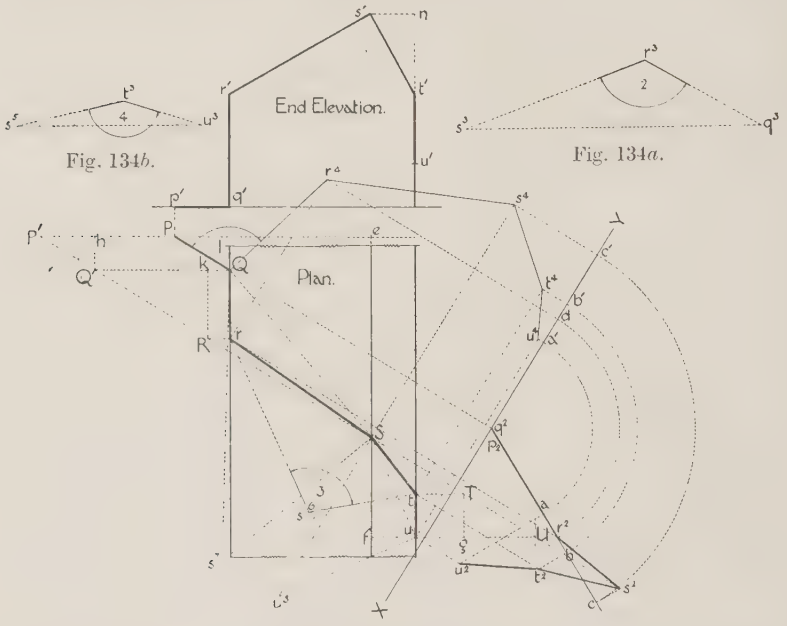


Fig. 134*b*.

Fig. 134*a*.

Fig. 134.

and end from a plane surface containing two of the sections, and to determine a bending plan.

Let u, u^1 be the projections of U. Develop the roof and wall surfaces about the plan of the ridge ef, so as to make eP^1 equal to the sum of the distances p^1q^1, q^1r^1, and r^1s^1, and make fU equal to the sum of the distances s^1t^1 and t^1u^1. Make PP^1 and uU each perpendicular to ef. Join P^1U to obtain the minimum length of pipe required. Make Ug equal to t^1u^1 and draw gT parallel to ef. Draw hQ^1 and hR parallel to ef, making P^1h equal to p^1q^1, and the perpendicular distance Q^1k equal to q^1r^1. Draw Q^1Q, Rr, and Tt perpendicular to ef. Then the plan of the pipe is $PQrStu$. Draw XY perpendicular to P^1U and determine an elevation on this ground line, as $p^2q^2r^2s^2t^2u^2$,

the heights above XY being equal to the heights of the corresponding points in the end elevation.

A straight line through p^2, q^2, r^2 is an edge view of the plane containing the two sections PQ and QR, and the perpendicular distances au^2, bt^2, and cs^2 are the required distances of the angles S and T and the end U from the bench surface containing PQ and QR, as explained in the previous problem. The bench plan may now be determined as follows: With p^2 as centre turn r^2, a, b, and c to a^1, d, b^1, and c^1. From r, S, t, and u draw parallels to XY to meet perpendiculars to XY from a^1, b, d^1, c^1 in r^4, s^4, t^4, and u^4; then $PQr^4s^4t^4u^4$ is the required plan over which the pipe may be bent. Procedure for bending. Take the required length of pipe P^1U and on it mark the positions Q^1R, S, and T. At Q^1 make the bend PQr^4 (marked 1). Set out the bending plan $PQr^4s^4t^4u^4$, and fix the two sections PQ and QR above their plans, leaving clearance enough to allow the section RS to be bent downwards an amount equal to cs^2. If S is made to stand directly over s^4 in addition to being depressed, the bend at R will be as required. The bend at S may now be made so as to bring T immediately over t^4 and the distance bt^2 above the surface containing PQ and QR. The bend at T must bring U directly above u^4, au^2 being the required height of U above the plane of PQR. For iron pipe work, where the elbows are required to be ordered or made, the actual bend at each point would be required; these may be determined as follows: The bend at Q is marked 1; for, since PQ is horizontal and Qr equals QR, and rr^4 is perpendicular to PQ, PQr^4 shows the two sections rabatted into the plane of the paper.

To determine the bend at R, as shown at 2, Fig. 134*a*. Join QS. Make Ss^7 perpendicular to QS and equal to the height of s^1; then Qs^7 is the real distance between Q and S. In any position (Fig. 134*a*), draw s^3q^3 equal to Qs^7, make s^3r^3 equal to RS, and q^3r^3 equal to Q^1R. Then $s^3r^3q^3$ is the real shape of the triangle whose plan is QrS and "2" is the required bend. The bend at S is determined by making ts^6 equal to ST, and rs^6 equal to RS; then, because r and t are equal in height, the required angle is "3" (Fig. 134). For the bend at T join Su and draw uu^5 perpendicular to it and equal to u^1n. Construct the triangle $u^3t^3s^5$ (Fig. 134*b*) with u^3s^5 equal to Su^5, u^3t^3 equal to TU, and t^3s^5 equal to ST; then "4" is the required angle.

For the purpose of relating as closely as possible the practical problems that arise out of constructional processes and the geometrical principles upon which the solutions are based, the following problems will be stated, first in practical terms and then as problems in descriptive geometry.

Wherever a problem of an unusual nature is met with in practice, the best method of solution is invariably found by mentally reducing the problem to the form of theoretical geometry. The following simple examples are given to illustrate how this may be done :

Problem 56. *Fig.* 135.—**The plan and elevation of one corner of a pitched roof are given. *ab*, a^1b^1 are the plan and elevation of the hip rafter, and *ce* is the plan of a jack rafter. *ap*, ap^2 are the plans, a^1p^1 the elevation of the wall plates. To determine the lengths and bevels for the hip and jack rafter.**

This, geometrically stated, is as follows :

Two intersecting straight lines *AB* and *CE* are shown by their projections ; determine the two lengths of the lines and the real angle between them. Determine also the angle between *AB* and a perpendicular to *HP* from *B*, and the inclination of *AB* to *HP*.

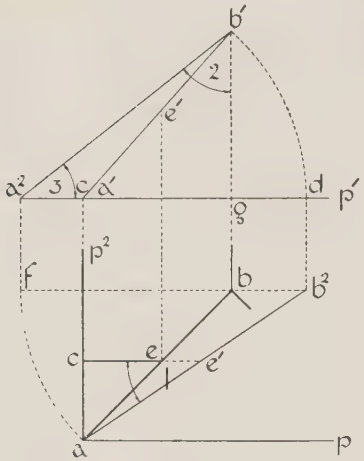


Fig. 135.

With *b* as centre, *ba* as radius, turn to *f* ; *bf* being parallel to a^1p^1. Produce p^1a^1 to meet a perpendicular to *bf* from *f* in a^2. Join a^2b^1 ; then a^2b^1 is the true length of *AB* (the hip rafter), the angle " 3 " is the inclination of *AB* to *HP* (the lower end bevel of the hip rafter), and the angle " 2 " is the angle between *AB* and a perpendicular to *HP* from *B* (the upper end bevel of the hip rafter).

With a^1 as centre, a^1b^1 as radius, turn to *d*. Produce *fb* to meet a perpendicular to a^1p^1 from *d* in b^2. Join ab^2 and produce *ce* to meet it in e^1. Then ab^2 is the real length of *AB* (the hip rafter), ce^1 is the true length of the line whose plan is *ce* (the jack rafter), and the angle " 1 " is the angle between *AB* and *CE* (the edge bevel for the jack rafter). The upper and lower face bevels for the jack rafters are the same as for the common rafters, and are the vertical and base angles of the right-angled triangle a^1b^1g.

Problem 57. *Fig.* 136.—**The plan and elevation of one corner of a turret roof, square in plan and ogee in elevation, are given. *ab* is the plan of the hip rafter, a^1h^1 . . . c^1 is the outline of the roof. Determine the true curved outline of the hip rafter. Or, a^1h^1 . . . c^1 is the elevation**

of a curved line lying in a plane whose horizontal trace is *ab* and whose vertical trace is bb^1. Rabat the plane about bb^1 to obtain the true shape of the curve.

Draw bk and a^1p^1 at right angles to bb^1. Select any number of points in a^1c^1, as $h^1g^1f^1e^1d^1$, and determine their plans. With b as centre, turn the plan of each point into bk and project perpendicularly to meet a horizontal from the elevation of the corresponding point to obtain a^2h^2 . . . c^2. A smooth curve through these points is the true shape of the curve or the outline of the hip rafter.

Problem 58. *Fig.* 137. **–The plan and section of a purlin intersecting a hip rafter in a pitched roof are given. Determine the face and edge bevels for the purlin PP^1. The horizontal wall plates are shown in**

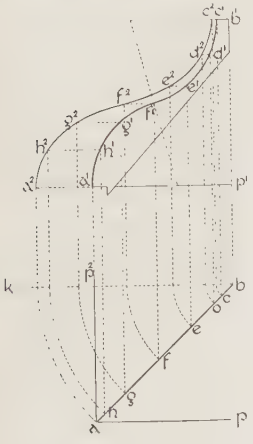


Fig. 136.

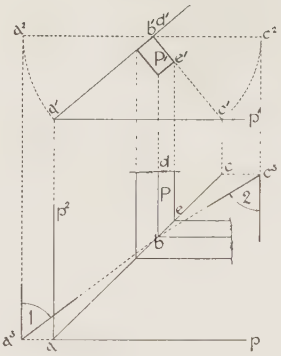


Fig. 137.

plan at *ap* and ap^2. The plan of the hip rafter is *ab*; the pitch of the roof is equal to the angle $b^1a^1c^1$.

Or, the projections of three intersecting straight lines *AB*, *BC*, and *BD* are given. Determine the angle between *BD* and *BC* and the supplement of the angle between *AB* and *BD*. When considered as a roof problem, c and c^1 are first determined by producing b^1e^1 to c^1 and dropping a perpendicular to meet ab, produced, in c.

Draw a parallel to a^1p^1 through b^1. With b^1 as centre, turn a^1 and c^1 to a^2 and c^2. Drop perpendiculars from a^2 and c^2 to meet parallels to a^1p^1 from a and c in a^3 and c^3. Join a^3b and c^3b. Then the angle c^3bd is the angle between BD and BC (the face bevel of the purlin), equal to the angle " 2 ". The angle dba^3 is the angle between AB and BD, and the angle " 1 " is its supplement (the edge of the purlin).

Problem 59. *Fig.* 138.—**The plan of the centre line of a wreathed handrail and the pitch of the two equally inclined straight rails connected by the wreath are given. Show three methods of developing the centre line, and show how the three developments are related by drawing circular arcs about the points that are common to two developments.**

Or, the square *abco* containing the quarter circle *ac* is the plan of a figure lying in the oblique plane whose horizontal trace is aa^2, and whose vertical trace is a^2d^1. Determine the elevation of the figure and determine its true shape by developing the plane : first, about its vertical trace ;

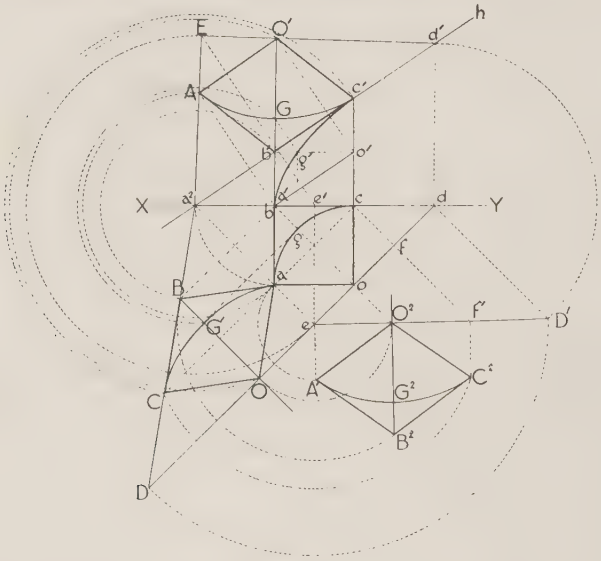


Fig. 138.

second, about its horizontal trace ; third, about an edge view of the plane.

Considered as a handrail problem in which the quarter circle *ac* is the plan of the centre line of the wreath, *ab* and *bc* being the plans of tangents at *A* and *C*, and b^1h the direction of the centre line of the upper straight rail, *a* must be turned about *b* to a^2. Draw a^2h inclined to *XY* at an angle equal to the pitch of the stairs. It will be shown later how this is done by developing the centre plane of the stair string.

Join a^2a. Then a^2a and a^2h are the traces of the plane of the plank, and the problem may now be considered as a general example of descriptive geometry.

Draw a parallel to a^2a through *o* ; in this particular example, where *AB* and *BC* are equally inclined, this parallel will pass through *b*.

Since bo is the plan of a line in the plane parallel to the ht (a^2a), it is a horizontal line, having b^1o^1 for its elevation, and g, the point where bo cuts the arc, projects to g^1.

Because a is in ht, a^1 is in XY. Draw a^1o^1; then because parallel lines project parallel, a^1o^1 is parallel to b^1c^1. The elliptic curve through g^1, tangent to a^1b^1 at a^1 and to b^1c^1 at c^1, is the elevation of the centre line of the wreath. Through o draw ed perpendicular to a^2a, and draw the perpendicular dd^1. Project e to e^1 and from a^1 and e^1 draw perpendiculars to a^2h to intersect circular arcs drawn about a^2 from a and e in A and E. Join a^2E and d^1E; a^2E will pass through A. Draw o^1O^1 perpendicular to a^2h, and join AO^1, Ab^1, c^1O^1, and b^1O^1. Then $AO^1c^1b^1$ is the development of $abco$, and is therefore a parallelogram. Draw g^1G perpendicular to a^2h. Then O^1G is the semi-minor axis and Ed^1 the direction of the major axis of the elliptic arc AGc^1; O^1b^1 being perpendicular to Ed^1. $AO^1b^1c^1G$ is the required development about the vertical trace.

To develop about the horizontal trace. Determine a^2, b^1, c^1, d^1, d, e, as before. With a^2 as centre, turn d^1 to meet de produced in D and join a^2D. Draw bB and cC parallel to dD and from B draw BO perpendicular to dD. Draw gG^1 parallel to bB and draw aB, aO, and CO. Then $aBCOG^1$ is the development about the horizontal trace, and it will be seen that the triangles a^2Ed^1 and a^2eD are congruent; and that corresponding points in the two triangles lie in the circumference of a circle drawn about a^2 as centre.

The third development is made about the edge view of the plane, as follows. Draw dD^1 perpendicular to ed and equal to dd^1; join eD^1 and draw cff^1 and oO^2 parallel to dD^1. Perpendicular to eD^1, draw eA^1, O^2B^2, and f^1C^2, making $eA^1 = ea$, $O^2B^2 = ob$, $f^1C^2 = fc$, and $O^2G^2 = og$. Join O^2A^1, O^2C^2, C^2B^2, and A^1B^2. The following pairs of distances are equal, as shown by circular arcs, centred at e: $ea = eA^1$; $eO = eO^2$; $eB = eB^2$; $eC = eC^2$; $eD = eD^1$.

CHAPTER X

PROBLEMS ON INTERSECTING INCLINED AND OBLIQUE PLANES

BEFORE the dihedral angle (greatest angle) between two plane surfaces can be determined, it is necessary to determine the projection and inclination of the line of intersection. In Figs. 139 to 144 the method of finding the projections of the line of intersection for several characteristic arrangements of planes is shown.

Consider Fig. 139, in which vth and $v^1t^1h^1$ are the traces of two oblique planes. Because ht and h^1t^1 intersect in a, a is actually a point in the line of intersection of the planes; and because a is in HP—being in the horizontal traces of the planes—its elevation will be a^1 in the ground line. Also, because the vertical traces intersect in b^1, b^1 is actually a point in the line of intersection of the planes, and since b^1 is in the vertical traces, it is in VP; therefore its plan will be in the ground line at b. The intersection of two planes is a straight line; and since the points a and b are both in HP, the straight line ab is the plan of the line of intersection of the given planes, and because a^1, b^1 are the elevations of two points in the line, the straight line a^1b^1 is the elevation of the line of intersection.

Problem 60. *Fig.* 140.—**Given the traces of an obtuse-angled oblique plane $v^1t^1h^1$ intersecting with an inclined plane vth, to determine the projections of the line of intersection.**

From b^1, the point of intersection of the vertical traces, drop a perpendicular on to XY at b. Let a be the point of intersection of the horizontal traces. Join a and b; then ab is the plan of the required line, and because the line is in the inclined plane, its elevation will be vt, the vertical trace of the plane.

Problem 61. *Fig.* 141.—**Given the traces of an acute-angled oblique plane $v^1t^1h^1$ intersecting the inclined plane vt, ht, to determine the projections of the line of intersection.**

The figure should require no explanation.

Problem 62. *Fig.* 142.—**Given the traces of two intersecting inclined planes, to determine the projections of the line of intersection.**

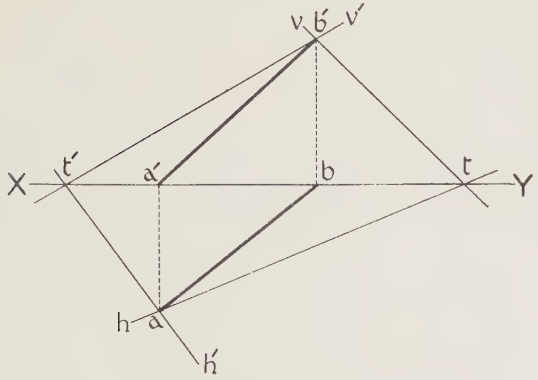


Fig. 139.

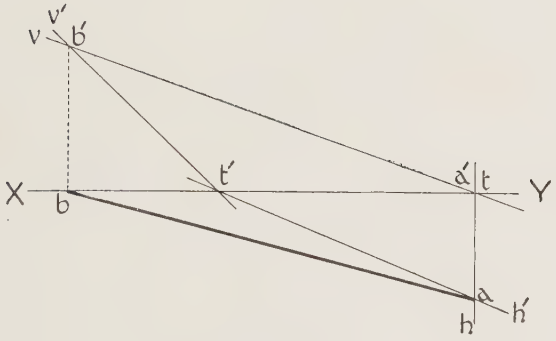


Fig. 140.

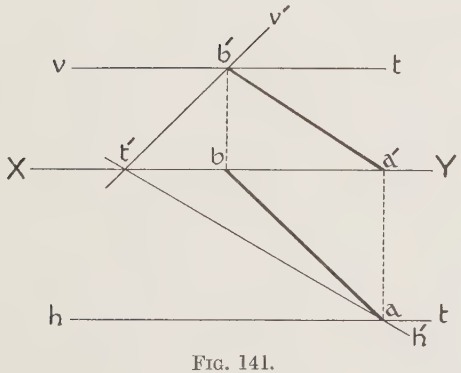


Fig. 141.

Let vt, ht be the traces of one plane, and v^1t^1, h^1t^1 the traces of the second plane.

Draw cd^1 perpendicular to XY to cut h^1t^1 in e, XY in d, and vt in f^1. With d as centre, turn e and c to e^2 and c^2. Draw e^2d^1 and c^2f^1 to intersect in g^2. From g^2 drop a perpendicular to n, and with d as centre, turn n to g. Then ab, and a^1b^1 parallel to XY through g and g^2, are the projections of the line of intersection of the planes. The base angles of the triangles dd^1e^2, ff^1c^2 are equal to the inclinations of the given planes to HP, and the vertical angles are equal to the inclinations of the planes to VP. The angle between e^2d^1 and c^2f^1 is the angle between the two given planes.

NOTE.—e^2d^1 and c^2f^1 are sections of the given planes, by a vertical plane on cd, turned about dd^1 into VP; g and g^1 being the projections of the point of intersection of the sections. And because

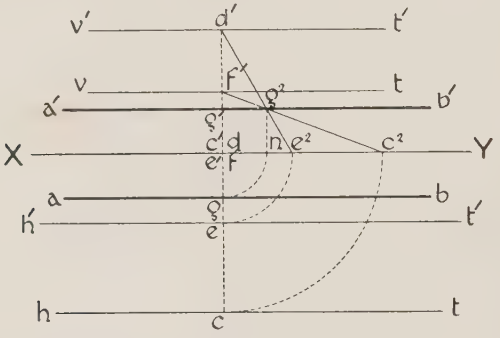


FIG. 142.

the traces are parallel, the line of intersection is parallel to both planes of projection, and therefore its projections are parallel to XY.

Problem 63. *Fig.* 143.—**Given the traces of two obtuse-angled oblique planes having their horizontal traces parallel, to determine the projections of the line of intersection.**

Let the intersection of the vertical traces be b^1. Draw b^1b perpendicular to XY. Draw ba parallel to ht and h^1t^1, and draw a^1b^1 parallel to XY. Then ab, a^1b^1 are the projections of the line of intersection. For since ht and h^1t^1 are parallel, the line of intersection is horizontal and its plan must be parallel to the horizontal traces of the planes containing it; and because the line is horizontal its elevation will be parallel to XY.

Problem 64. *Fig.* 144.—**Given the traces of two obtuse-angled oblique planes vth, $v^1t^1h^1$, to determine the projections of the line of intersection.**

In this case it is only necessary to note that the projections of the line of intersection fall outside the space enclosed by the traces.

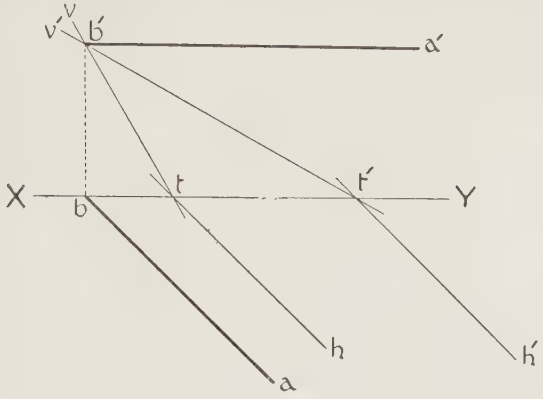


Fig. 143.

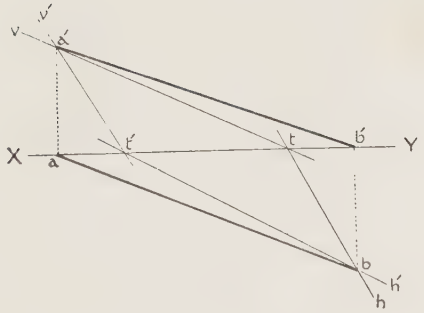


Fig. 144.

Problem 65. *Fig.* 145.—**Given the traces of two intersecting oblique planes, all intersecting in the ground line, to determine the projections of the line of intersection.**

Let vth and $v^1t^1h^1$ be the given traces, and a be the point in XY in which they intersect. Draw h^2t^2 to make any angle other than 90° with XY, and draw $t^2\,V^2$ perpendicular to XY.

From p and q, where h^2t^2 cuts ht and h^1t^1, draw perpendiculars to meet XY in p^1 and q^1. Join p^1, r^1, and $q^1\,s^1$; r^1s^1 being the points of intersection between v^2t^2 and the given vertical traces. From b^1, the point of intersection of p^1r^1 and q^1s^1, drop a perpendicular to meet h^2t^2 in b. Then straight lines through a, b and a, b^1 are the required projections. For $v^2t^2h^2$ are the traces of a vertical plane which cuts the given planes in two straight lines whose projections are p^1r^1 and q^1s^1,

and since b^1 is the elevation of the point of intersection of these two lines, it is the elevation of a point in the line of intersection of the two given planes. Because b^1 is in the vertical plane $v^2t^2h^2$, its plan will be

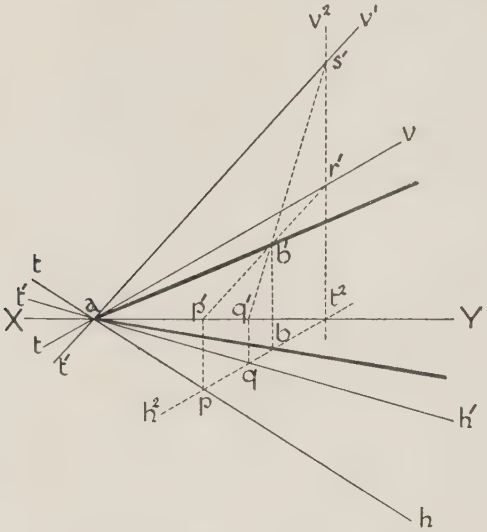


Fig. 145.*

in h^2t^2. Since also all the traces pass through a, a is an actual point in the line of intersection of the given planes. Then ab and ab^1 are the required projections.

Dihedral Angle.—In Fig. 146 a pictorial view of a triangular pyramid is shown containing the triangular section DEF made by a plane perpendicular to the line $A\,V$. The vertex angle α of the triangle is the angle between the faces $AC\,V$ and $AB\,V$. Because the plane of the section is perpendicular to $A\,V$, every line in the plane is perpendicular to $A\,V$; therefore the lines DF and DE are each at right angles to $A\,V$, and since $A\,V$ is the line of intersection of the planes $AC\,V$, $AB\,V$, the angle between the lines DF, DE is the angle between the planes. Let Av be the projection of $A\,V$ on the plane ABC; then because the plane of the triangle DEF is perpendicular to $A\,V$, FE is at right angles to Av. Also the plane $A\,Vv$ is perpendicular to the plane DEF, and the line of intersection of the two planes is DG; therefore DG is perpendicular to both FE and $A\,V$. There are thus five right angles about the triangle DEF and the two lines $A\,V$ and Av, indicated at 1, 2, 3, 4, and 5 in the figure. It should be noted that F and E are in the traces of the planes $AC\,V$, $AB\,V$ on ABC, and that the angle $vA\,V$ is the inclination of the line of intersection $A\,V$ to the

plane *ABC*. It is necessary to consider the figure carefully in connection with the following problems on the dihedral angle, since the

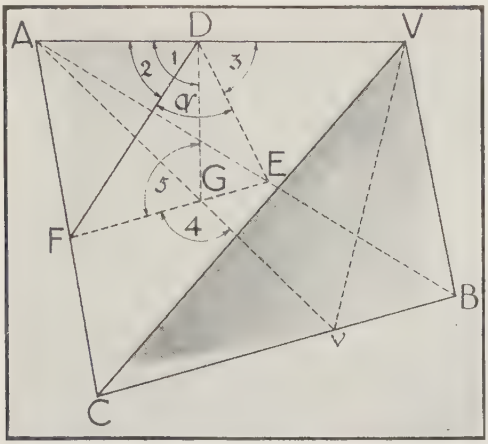


Fig 146.

solution in each case is dependent upon the determination of a triangle related to the planes involved as is *DEF* to *AC V* and *AB V*.

Now consider Fig. 147, in which the two planes *vth*, $v^1t^1h^1$ are given, and the angle between the planes is required. The projections of the line of intersection *ab*, a^1b^1 are determined as in previous problems. Draw *bB* perpendicular to *ab* and equal to bb^1, and join *a*, *B*. Then *aB* is the true length and the angle *baB* is the inclination to *HP* of the line of intersection. Through any point *E* in *ab* draw a perpendicular to *ab* to meet the horizontal traces of the planes in points *C* and *D*. Draw *EF* perpendicular to *aB* and with *E* as centre, *EF* as radius, cut *ab* in F^1. Join F^1, *C* and F^1, *D*; then *α*, the angle between CF^1 and DF^1, is the angle between the given planes. Compare with Fig. 146,

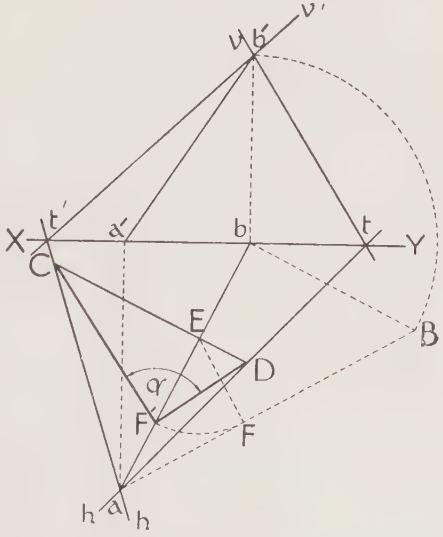


Fig. 147.

and note as follows: The triangle *abB* in Fig. 147 is the rabatment of the vertical plane containing the line of intersection about its plan, and is the equivalent of the triangle *Av V*, Fig. 146, *GD* in the latter figure corresponding to *EF* in Fig. 147. Also *FE*, Fig. 146, and *CD*, Fig. 147, are corresponding lines, and since, in Fig. 147, EF^1 is perpendicular to *CD* and equal to *EF*, the triangle CF^1D is related to the given planes precisely as is the triangle *FDE*, Fig. 146, to the planes *AC V* and *AB V*.

Analysis of the Dihedral Angle by Development.

Fig. 148.—The work in this exercise provides an excellent form of revision for most of the preceding matter on descriptive geometry.

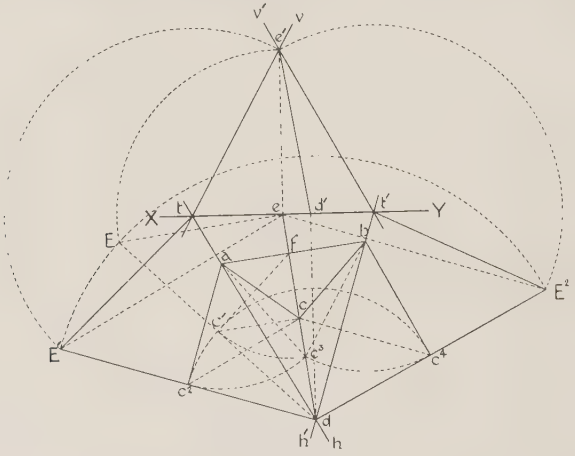


Fig. 148.

Two intersecting oblique planes *vth*, $v^1t^1h^1$ are given, and it is required to find the dihedral angle and to develop the planes to show that this angle is contained by two lines, one in each plane, each line perpendicular to the line of intersection of the planes.

Determine *de*, d^1e^1, the projections of the line of intersection, and make *eE* equal to ee^1 and perpendicular to *de*. Then the angle *edE* is the inclination of the line of intersection to *HP*.

Draw *ab* to intersect *ed* perpendicularly in any point *f*, *a* and *b* being in the horizontal traces of the given planes. Draw fc^1 perpendicular to *dE*, and draw c^1c perpendicular to *ed*. Join *a*, *c* and *b*, *c*. Then *abc* is the plan of the section of the given planes made by a plane perpendicular to the line of intersection, *ab* being the horizontal trace of the cutting plane. With *f* as centre, fc^1 as radius, cut *de* in c^3. Join

a, c^3 and b, c^3, then the triangle ac^3b is the true shape of the section and the angle ac^3b is the angle between the planes.

From e draw perpendiculars to ht and h^1t^1 to intersect circular arcs drawn about t and t^1 through e^1 in E^1 and E^2. Join E^1 to t and d and join E^2 to t^1 and d; then the triangles E^1td and E^2t^1d are the given planes developed into HP, and since E^1d, Ed, and E^2d are developments of the line of intersection of the given planes, they are equal in length, and the points E, E^1, and E^2 lie in the circumference of the same circle, drawn about d as centre.

Draw cc^2 and cc^4 perpendicular to ht and h^1t^1 respectively, and join a, c^2 and b, c^4. Then ac^2 is perpendicular to E^1d and bc^4 is perpendicular to E^2d, and are the developments of the two lines containing the dihedral angle, each being then, as required, perpendicular to the line of intersection of the given planes. But ac^3, bc^3 are also developments of the lines containing the dihedral angle, therefore ac^3 equals ac^2 and bc^3 equals bc^4, as shown by the circular arcs drawn about the centres a and b. Also, because c^1, c^2, and c^4 are developments of the same point, and the lines containing the developments have corresponding extremities in the common point d, dc^2, dc^1, and dc^4 are equal and are in the circular arc, centre d.

This diagram shows the setting out for a useful paper model that may be cut and folded as follows: Cut cleanly along the lines te^1, t^1e^1, dE^1, t^1E^2, dE^2, ac^3, and bc^3. Now fold along the lines tt^1, td, t^1d, and ab so as to bring the points e^1, E^1, and E^2 together. Then c^2 and c^4 will coincide, and c^3 may be turned up about ab to c^2 and c^4, so as to bring the lines ac^2, ac^3 together and the lines bc^3, bc^4 together.

Problem 66. *Figs.* 149 *and* 150.—**Given the traces of any two intersecting planes, to determine the angle between the planes.**

The two examples solved in Figs. 149 and 150 are chosen because of their peculiarities, which are sometimes regarded as difficulties. Actually there can be no degrees of difficulty between one case and another, since the procedure for finding the angle between two planes is always the same, and is as follows: Determine a projection and the inclination of the line of intersection of the planes to the plane containing the projection.

Draw any perpendicular to the projection to meet the traces of the planes on the plane containing the projection. About the point in which the perpendicular cuts the projection, draw a circular arc tangent to the line of inclination to cut the projection in two points. Join either of these points to the points in which the perpendicular meets the traces; then the angle between these lines is the required angle.

Consider Fig. 149, in which an obtuse-angled oblique plane *vth*

intersects an inclined plane $v^1t^1h^1$. *ab* is the plan, and the angle *bag* the inclination of the line of intersection. Perpendicular to, and from any point *f* in *ab*, draw a straight line to cut the horizontal traces in points

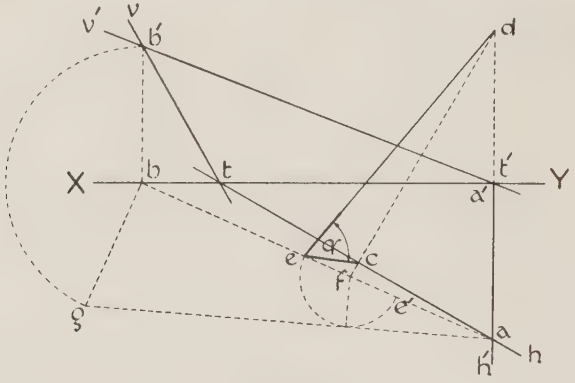


Fig. 149.

c and *d*. With *f* as centre, draw a semicircle tangent to *ag* to cut *ab* in points *e* and e^1. Join *e*, or e^1, to *c* and *d*; then the angle *ced* (α) is the required angle.

Second case: Fig. 150. The traces of two oblique planes, *vth*, $v^1t^1h^1$, all intersect in the ground line; determine the angle between the planes.

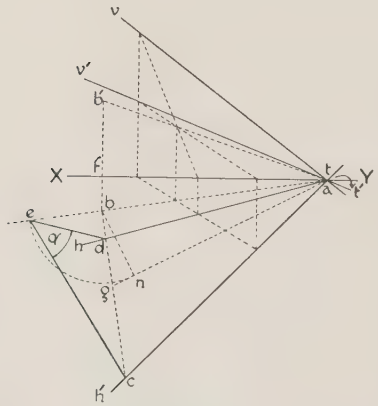


Fig. 150.

Determine the projections of the line of intersection *ab*, ab^1, as explained in problem 65 (Fig. 145). From any convenient point *b* in *ab*, draw bb^1 perpendicular to *XY* and *bdc* perpendicular to *ab*, *d* and *c* being points in the horizontal traces of the given planes. Make *bg* equal to fb^1 (the height of *B*). Draw *ag*, and perpendicular to it draw *bn*. With *b* as centre, *bn* as radius, cut *ab* produced in *e*. Join *e*, *d* and *e*, *c*; then the angle *ced* (α) is the required angle.

It should be carefully noted that in every case the constructions are the same, and that only the relative positions of the lines vary with alterations in the directions of the planes.

Problem 67. *Fig.* 151.—**A regular octagonal prism, *abcdefgi*,**

standing with its axis vertical, is intersected by an oblique plane *vth*. Determine the angles made by the plane with the vertical faces of the prism.

This problem is arranged to present the dihedral angle under conditions similar to those met with in many practical cases, and to show how the lines may be extended to obtain results sufficiently large for ensuring a reasonable degree of accuracy. The elevation of the prism is not shown, since it is not required in the solution.

First consider the face of the prism whose plan *ab* is perpendicular to *XY*. Produce *ab* to cut *XY* in *j*, and *vt* in j^1. Make *jJ* equal to jj^1 and join *J*, *a*. Draw *jo* perpendicular to *Ja* and make *jO* equal to *jo*. Join *tO*. Then the angle *jOt* (1) is the angle between the plane *vth* and the face *ab* of the prism. For *aj* is the plan of the line of intersection between the given plane and the vertical plane containing the face *ab*, and since the line of intersection has one extremity in *ht* at *a*, and the other in *vt* at j^1, the true length is *aJ* and its inclination is *jaJ*. Also *jt* is perpendicular to the plan *aj*, and meets the horizontal traces of the two planes in *t* and *j*. Then, since *jo* is at right angles to *Ja*, the rabatted line of intersection, the triangle, whose vertical angle is equal to the required angle, has *jt* for its base and *jO* for its perpendicular.

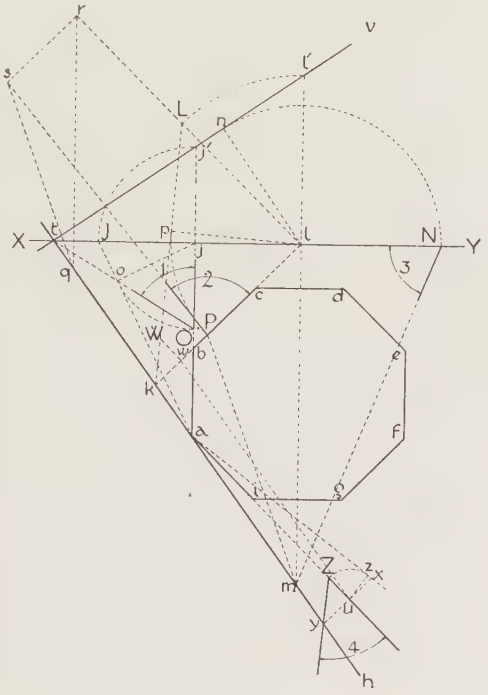


Fig. 151.*

Now consider the face of the prism whose plan is *bc*.

Produce *bc* to meet *ht* in *k*, and *XY* in *l*; then *kl* is the plan of the line of intersection between the given plane *vth* and the vertical plane containing the face *bc*. Draw ll^1 perpendicular to *XY* and *lL* perpendicular to *kl*, and make *lL* equal to ll^1. Join *lk*; then the inclination

of the line of intersection is *Lkl*. Draw *pl* perpendicular to *Lk* and make *lP* equal to *lp*.

The ordinary procedure is now to produce the line *lL*, perpendicular to *lk*, to meet *ht*; but owing to the large angle between *lk* and *ht*, this is impracticable and the following construction must be resorted to:

Join *P* and *l* to any point *m* in *ht*. From any second point *q* in *ht*, draw a parallel to *lm* to meet *lL* produced in *r*. Draw *rs* parallel to *lk* and *qs* parallel to *Pm*. Join *sP*; then *sPl* (2) is the required angle. For since the triangles *lmP*, *rqs* are similar, with corresponding sides parallel, *ht*, *Ps*, and *lr* would, if produced, meet in a common point.

The angle between the face *cd* and the given plane is equal to the angle between the vertical plane and the given plane, *cd* being parallel to *XY*. Therefore let *lm* be perpendicular to *XY* and *ln* perpendicular to *vt*; make *lN* equal to *ln* and join *Nm*. The angle *lNm* (3) is the one required.

Finally, to find the angle made by the given plane and the face *ai*. The angle between *ai* and *ht* being comparatively small, *ai* is produced to any convenient point *u*. Draw *uw* parallel to *ht* and draw *wW* perpendicular to *kl*; then *wW* is the height of the horizontal line *uw*. Draw *xy* through *u* perpendicular to *au* and make *ux* equal to *wW*. Join *ax* and draw *uz* at right angles to *ax*. Make *uZ* equal to *uz* and draw *Zy*. Then the angle *uZy* (4) is the required angle.

Parallel faces of the prism make equal angles with the given plane.

Practical Problems involving the Determination of Lines of Intersection and the Dihedral Angle.

Problem 68. *Fig.* 152.—**To determine the bevels required in making the mould for the hip tiles and backing the hip rafter for a roof whose plan and inclination are given.**

The hip rafter in a roof is the member which extends from the top of the wall to the ridge of the roof at the intersection of two sloping roof surfaces. When the member is of timber, it is at first rectangular in section, and when in position its faces are in vertical planes and its upper edge is shaped so as to have half the surface in each roof slope. The bevel required for shaping the edge of the timber is called the backing bevel, and is equal to half the angle between the two roof slopes that intersect along the hip rafter. The bevel is applied from each face of the hip, as shown in the figure, to produce the full dihedral angle (1). The same angle is required for the hip tile mould, together with the angle marked (2). The hip tile, whose position on the roof is indicated at *m*, *n*, Fig. 152, is made up of two flat sections set to make

the angle (1) with each other, each section being similar to the figures *fghd*, so as to have the edges *fg* and *hd* horizontal, and *gh* perpendicular to the ridge.

To make the mould, the two sections are prepared to the required shape and then mitred together to contain the dihedral angle.

The bevels are determined as follows: Let *DP* and *DQ* represent the lower horizontal edges of the two roof slopes, *De* the plan of the centre line on the upper edge of the hip rafter, *ek* the plan of the centre line on the upper edge of the horizontal ridge, and the angle *CAB* the slope of the roof; *AC* being perpendicular to *PD*. Then *CB* is the

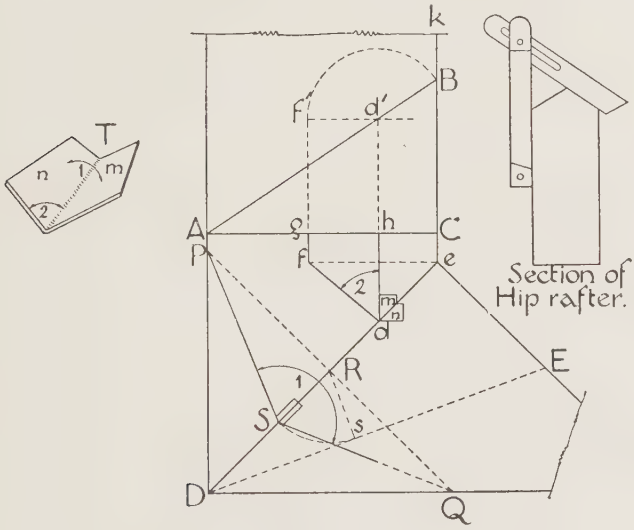


Fig. 152.

difference in the heights of the upper and lower ends of the hip rafter, and if *eE* is drawn perpendicular to *De* and equal to *CB*, *DE* is the length and *EDe* the inclination of the line of intersection between the roof slopes. Geometrically stated, we get *DP* and *DQ* as the horizontal traces of two planes, *De* the plan of the line of intersection of the planes, and *eDE* the inclination of the line of intersection. Proceeding as in previous examples, draw *PQ* to cut *De* perpendicularly in any point *R*. Draw *Rs* perpendicular to *DE*. Make *RS* equal to *Rs* and join *S*, *P* and *S*, *Q*. *PSQ* (1) is the angle between the roof slopes, or planes.

The angle (2) is the real angle between the horizontal line *DP* and the line of intersection *DE*, and may be found as follows: From any point *d* in *De* draw dd^1 parallel to *DP*, cutting *AC* perpendicularly in *h*. Draw f^1d^1 parallel to *AC*. With d^1 as centre, turn *B* to f^1. From f^1

parallel to DP and from e parallel to AC draw lines to intersect in f. Join f, d; then the angle hdf (2) is the required angle for shaping each section of the hip tile mould. A sketch of the mould or tile is shown separately at T.

In Fig. 152a the backing bevel for the hip rafter in a roof over an irregular plan is determined. The lettering accords with that

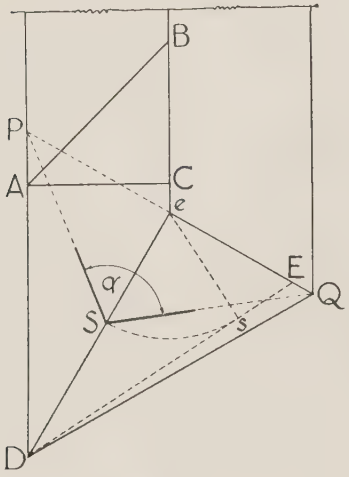


Fig. 152a.

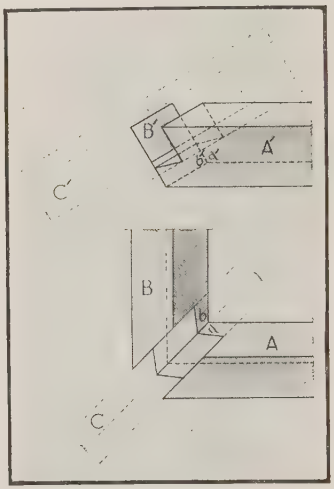


Fig. 153.

in the last problem, but R is dispensed with, since PQ is in the same straight line with eE. The figure should require no further explanation.

The Geometry of the Roof Purlin.

The horizontal members in a roof which serve as intermediate supports to the rafters are called purlins. The purlins are supported by the hip rafters, the joint between the two purlins and the hip being arranged as shown in Fig. 153. The purlins are shown in plan at A and B, and in elevation at A^1 and B^1, the hip rafter is shown dotted at C and C^1. Usually the depth and position of the purlin are such as to cause the lower edge to project below the lower edge of the hip, and a lip must be formed on the end of each purlin so that the lower edges of the two purlins may meet in a mitre joint. The plane of the mitre is in the vertical plane containing the longitudinal mid-section of the hip rafter. In the figure the lower edges of the purlins A and B are continued beyond the faces of the hip so as to form the projecting lips a and b. The lower edge of the hip rafter is a plane surface at right angles to the vertical faces, and the upper surfaces of the two lips

must lie in this plane. This arrangement provides an exceptionally interesting geometry model of intersecting planes and lines and should be carefully studied for its general, rather than its particular, applications.

An isometric view of the purlin end is shown in Fig. 154, and it will be seen that there are ten angles indicated by numbers. It is proposed to analyse these angles and to show the natures of the relationships existing between them.

First consider the positions of the surfaces A, B, C, D, and E in the roof. The surface A, called the edge of the purlin, is parallel to the plane of the roof surface; therefore, since the cross section of the

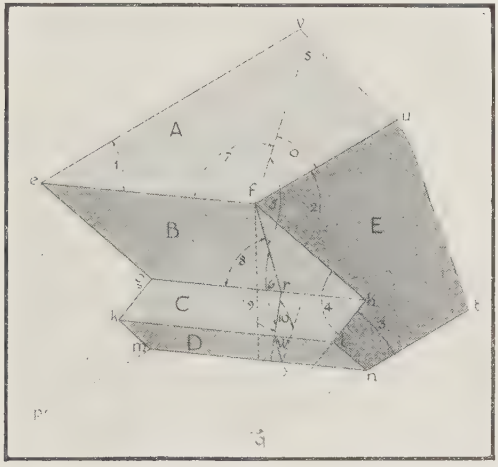


Fig. 154.

purlin is rectangular, the surface E, called the face of the purlin, is perpendicular to the roof surface. The surface B lies in the vertical plane of the face of the hip rafter. The surface C is in the plane of the under edge of the hip rafter. The surface D is parallel to B, being in the vertical plane of the centre section of the hip.

The angle 1 is called the edge bevel. The angle 2 is called the face bevel. The angle 3 is called the lip bevel.

The lines sf, fr, rw, wy are the traces of a plane, perpendicular to the four parallel lines ef, gh, kl, and mn on the faces A, B, C and D; therefore the angles 7, 8, 9 are right angles. Since the plane containing the points s, f, r, w, and y is perpendicular to ef, it is perpendicular to the plane A containing ef; and because the planes A and E are at right angles, the line of intersection between the plane E and the plane

sfrw is perpendicular to the line *fu*, the line of intersection between planes *A* and *E*. Let *rw* and *hl* be produced to meet in *q*; then the line joining *f* and *q* is the line of intersection between the planes *E* and *sfrw*, and is at right angles to *fu*. The angle between the planes *E* and *sfrw* is the angle *O*, and since *sf* is perpendicular to *ef*, *O* is the complement of the edge bevel (1).

It has already been mentioned that the plane *A* is parallel to the roof plane and that the plane *B* is in the vertical plane of the face of the hip rafter; then, since the vertical centre plane of the hip bisects the angle between the intersecting roof planes, and the face plane of the hip is parallel to the centre plane, the angle between the surfaces *A* and *B* (5) is half the dihedral angle between the roof planes. Because the surface *B* is in the face, and the surface *C* is in the edge of the hip rafter, and the face is perpendicular to the edge, the angle between the surfaces *B* and *C* (6) is a right angle. Also, because the lines *fs* and *qy* are parallel and the angle 6 is a right angle, the angle between the lines *qr* and *qy* (10) is the complement of half the dihedral angle (5).

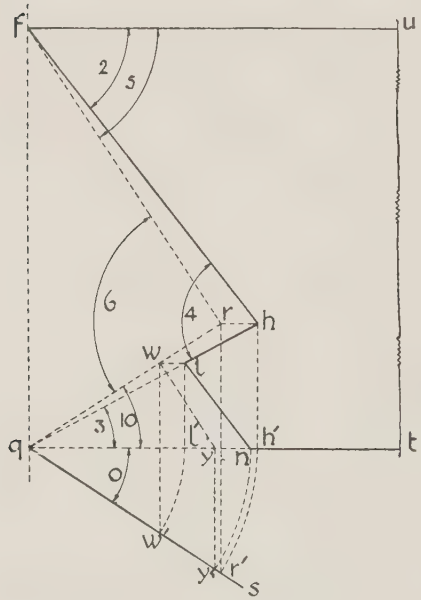


Fig. 155.

It will be seen then that if the angles 5, 6, and 10 (5 being half the dihedral angle, 10 its complement, and 6 a right angle) be set out on a plane and projected perpendicularly to this plane on to a second plane, the angle between the two planes being the complement of the edge bevel, the angle 5 projects into the face bevel 2, the right angle 6 into the angle 4, and the angle 10 into the lip bevel 3. The method of projection is shown in Fig. 155; corresponding angles in Figs. 154 and 155 are denoted by corresponding numbers. Refer to Fig. 155. Let the angle 5 equal half the angle between the roof slopes, the angle 6 be a right angle, and let *qt* be parallel to *fu*; then the angles 5 and 10 are complementary.

Make *rw* equal to half the thickness of the hip rafter and draw *wy*

parallel to *fr*. Produce *rw* to meet a perpendicular to *fu* from *f* in *q*. Draw *qs* to make the angle (*O*) with *qt* equal to the complement of the edge bevel of the purlin. Draw perpendiculars to *qt* from *w*, *y*, and *r* to meet *qs* in points w^1, y^1, and r^1. About *q* as centre, turn w^1, y^1, and r^1 to cut *qt* in l^1, *n*, and h^1. Draw perpendiculars to *qt* from l^1 and h^1 to meet parallels to *qt* from *w* and *r* in *l* and *h*. Join *f*, *h*; *h*, *l*; and *l*, *n*, to obtain the angles 2, 4, and 3. Then 2 is the face bevel and 3 the lip bevel, and the lines *fh*, *hl*, *ln* are the cutting lines for the purlin face.

The complete set of bevels for the purlin end may be determined as follows:

Fig. 156.—The angles in the figure are numbered to correspond with the two preceding figures.

Let *ap* and ap^1 be the horizontal traces of the two roof planes, *ar* the plan of the line of intersection of the planes, and the angle $p^2a^1r^1$ the inclination of each plane to *HP*.

The face and edge bevels (1 and 2) of the purlin are determined by a method already explained in problem 58, in which the problem was stated both as a roof problem and as a problem on intersecting lines. It will now be stated and worked as a problem on lines of intersection of planes.

Consider *eb* and e^1b^1 as the projections of the trace of an inclined plane containing, or parallel to, the edge plane of the purlin, on the vertical centre plane of the hip; then the angle between this trace and any horizontal line in the inclined plane is the angle required for the edge bevel of the purlin. Since *ee*^{1} and *bb*^{1} are parallel to the horizontal trace of this plane, they are horizontal lines. Then if the line of intersection is turned into a horizontal plane containing one of these lines, the real angle will be seen in plan. Also, *cb* and c^1b^1 are the projections of the trace of the inclined plane containing, or parallel to, the plane of the face of the purlin, on the vertical centre plane of the hip, and *bb*^{1} and *cc*^{1} are horizontal lines in the inclined plane. If the trace *bc*, b^1c^1 is turned about one of these horizontal lines into a horizontal

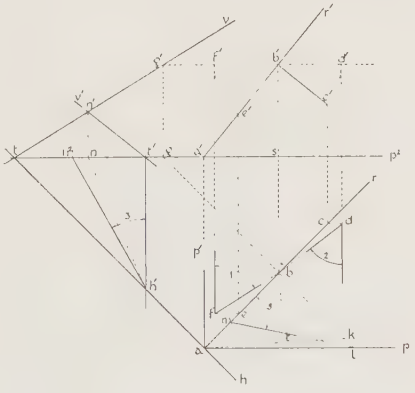


Fig. 156.*

position, the real angle between the trace and the line will be seen in plan, and is the required face bevel for the purlin. Draw the horizontal line f^1d^1 through b^1 and make b^1f^1 equal to b^1e^1 and b^1d^1 equal to b^1c^1. Draw f^1f and d^1d parallel to bb^1 and draw ef and cd perpendicular to bb^1. Join f, b and d, b; then the angle f^1fb (1) is the edge bevel, and the angle between bd and dd^1 produced is the face bevel (2).

The angle 5 is half the dihedral angle between the roof planes, or the angle between one of the roof planes and the vertical centre plane of the hip, determined as follows :

Draw bl perpendicular to ar, l being in the horizontal trace ap. Make bk equal to sb^1 and join k, a; then the angle bak is the inclination of the line of intersection. Draw bt perpendicular to ak, and with b as centre, turn t to m. Join m, l; then the angle bml (5) is half the dihedral angle between the roof planes, or the angle between one of the planes and the vertical centre plane of the hip.

The lip bevel (3) is the angle between the horizontal trace of the face plane of the purlin and the line of intersection between this plane and the plane containing the lower edge of the hip rafter.

Since the faces of the hip are in vertical planes and the edge is at right angles to the faces, ht drawn through a perpendicular to ar is the horizontal trace of a plane parallel to the lower edge of the purlin. Draw bg parallel to ht and from g erect a perpendicular to meet a horizontal line through b^1 in g^1; then vt, through g^1, is the vertical trace of the plane parallel to the edge plane of the hip rafter. Through any point n^1 in the vertical trace vt, draw n^1t^1 parallel to b^1c^1 and draw h^1t^1 perpendicular to tt^1; then $v^1t^1h^1$ are the traces of an inclined plane parallel to the face plane of the purlin and nh^1 is the plan of the line of intersection between the two planes vth, $v^1t^1h^1$. With t^1 as centre, turn n^1 to n^2, and draw n^2h^1. The angle $t^1h^1n^2$ (3) is the lip bevel for the purlin.

Geometry of a Voussoir in a Skew Arch.

Figs. 157 *and* 158.—In Fig. 157 the elevation, half plan, and half inverted plan of a skew elliptical arch are shown. The joint lines on the face of the arch are normals to the curve and are horizontal across the arch soffit. It is proposed to treat the voussoir SS^1 as a geometrical model and to show how the angles between the intersecting lines and surfaces, and the true shapes of the surfaces, may be determined. The plan and elevation of the voussoir are redrawn in Fig. 158, and the geometric treatment shown in the figure is not restricted to the comparatively simple requirements of the mason for shaping the material.

Referring to Fig. 158, first consider the block of stone from which

the voussoir is to be worked to have *kdlg* for its plan and $k^1d^1l^1m^1$ for its elevation. Then, since the two faces of the voussoir *ACEIG* and

Fig. 157.

BDFJH are in the faces of the block, the true shape of each of these faces seen in elevation at $a^1c^1e^1i^1g^1$ is the shape of the required face moulds, and the angle made by each horizontal arris of the voussoir with the faces is the plan angle, *egh*. The horizontal surface *ABCD* shows true shape in plan, and is of course at right angles to the perpendicular faces *ACEIG*, *BDFJH*, and *CDFE*.

Draw *gu* and *dt* perpendicular to *gh* and *de*, and on *gu* as a ground line project a second elevation of the voussoir (gg^2 equal to m^1g^1, and similarly for other points). Then, because $a^2c^2e^2i^2g^2$ is the true shape of a section perpendicular to the parallel horizontal arrises *AB*, *CD*, *EF*, *IJ*, *GH*, the angles between the lines in the section are the real angles between the faces; thus, the angle $c^2a^2g^2$ is the dihedral angle between the faces *ABCD* and *ABGH*, and the angle $c^2e^2i^2$ is the dihedral angle between the faces *DCEF* and *FEIJ*.

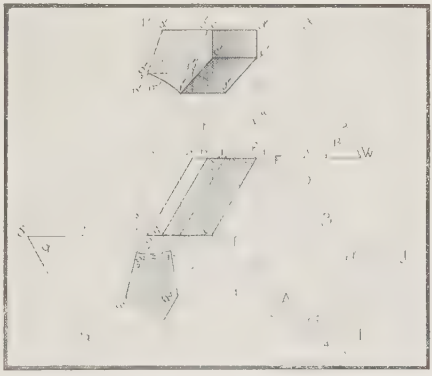


Fig. 158.*

If the voussoir were required to be worked from the block of stone, whose plan is *gtud*, depth equal to m^1k^1, the moulds required for the

end planes of the block, *gu*, *td*, would be of the same shape as the section, $a^2c^2e^2i^2g^2$. Now suppose the block to have been worked from these moulds into the form of the prism shown at *tdgu* in plan; then the lines required on the rectangular faces of the prism to obtain the vertical faces *ACEIG* and *BDFJH* of the finished voussoir may be found by development as follows: On *gu* produced determine the points 0, 1, 2, 3, 4, and 5, making $i^20 = i^2e^2$, $01 = e^2c^2$, $12 = c^2a^2$, $23 = a^2g^2$, $34 = g^2z$, and $45 = zi^2$. From *c* and *d* draw perpendiculars to *cd* to meet a parallel to *cd* from 0 in *E* and *F*. Join *i*, *E* and *j*, *F*; then *FEij* is the true shape of the face *IEFJ*. The remaining faces are developed in a similar manner, the lines *GI* and *HJ*, in the development of the face *GHJI*, being developments of the curved lines *GI* and *HJ*.

To determine the dihedral angle between the vertical face *ACEIG* and the sloping faces *ABGH* and *EFIJ*. Produce a^1g^1 to n^1 and from n^1 drop a perpendicular to meet *ag* produced in *n*. Draw aa^3 perpendicular to *na* and equal to m^1k^1. Through any point *r* in *an* produced, draw *qr* perpendicular to *an*; *nq* being parallel to *gh*. Draw *rs* perpendicular to a^3n produced, and with *r* as centre, *rs* as radius, cut *nr* produced in *S*. Draw *Sq*; then the angle *rSq* (α) is the dihedral angle between the faces *ABGH* and *ACEIG*. For *nq* and *nr* are the horizontal traces of the planes containing the faces, and *rns* is the inclination of the line of intersection between the faces. Then, for reasons already explained, *qSr* is a section of the two planes by a third plane perpendicular to the line of intersection, and is the greatest angle between the planes.

The dihedral angle between the faces *BDFJH* and *EFJI* is shown at β, and is determined as follows: Produce *ij* and *jf*, the horizontal traces of the planes of the faces, to any convenient distances, and from any point *y* in *jf* produced draw a perpendicular to *jf* to meet *ji* produced in *x*. Make *fv* perpendicular to *jf* and equal to l^1f^1 and draw a straight line from *j* through *v*; then *fjv* is the inclination of the line of intersection. Draw *yw* perpendicular to *jv* and with *y* as centre, *yw* as radius, cut *jf* produced in *W*. Join *Wx*; then *yWx* is the required angle. The angles between each pair of intersecting straight line arrises, and between each pair of intersecting faces, are now shown in the figure.

Geometry of Joints in Splayed Work.

Figs. 159 *and* 160.—Fig. 159 may be considered as the corner of a splayed lining, or the corner of a box having splayed sides. When considered as the former, the upper section of the diagram would be the plan instead of the elevation. The geometry is the same in both cases.

Let *abedhgfcs* be the plan of one corner of a box having its sides inclined to the horizontal at an angle equal to $k^1d^1a^1$, and intersecting in a vertical mitre plane.

The lower edges of the sides are to be in one plane, the upper edges at right angles to the faces. It is required to find the dihedral angle between the sides, and the face and edge bevels for mitring.

Draw *dp* perpendicular to *ad* and equal to ra^1. Let *pd*, produced where necessary, cut the horizontal edges *ab* and *ag* in points *n* and *m*. Draw *dq* at right angles to *ap*, and with *d* as centre, turn *q* to *Q*. Join *Q*, *m* and *Q*, *n*. Then the angle *mQn* (1) is the dihedral angle between the sides.

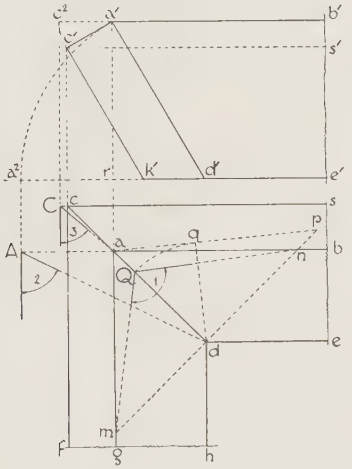


Fig. 159.

Note.—Since *ab* and *ag*, the selected horizontal traces, containing *m* and *n*, are the upper edges of the two face planes, the difference between the heights of *a* and *d* is set out from the lower end *d* of the line of intersection.

To find the face bevel, make d^1a^2 equal to d^1a^1 and drop a perpendicular from a^2 to meet *ba* produced in *A*. Join *A*, *d*; then the angle (2) is the required face bevel. To find the edge bevel, produce b^1a^1 to c^2, making a^1c^2 equal to a^1c^1. Drop a perpendicular from c^2 to meet *sc* produced in *C*. Join *Ca*; then the angle (3) is the required edge bevel.

The second example, Fig. 160, shows the sides tongued and grooved, instead of mitred together, and to give additional support to the tongue the grooved side is allowed to run on a short distance beyond the outer face of the tongued side. All other conditions are the same as in the

previous example. It is required to draw the plan and elevation of the joint and to determine the angle between the sides of the groove and the face of the grooved side. Draw the elevation with the surface c^1k^1 of the projecting end in the plane of the upper edge of the tongued side. Draw e^1p^1 parallel to a^1r^1 to represent the desired thickness of

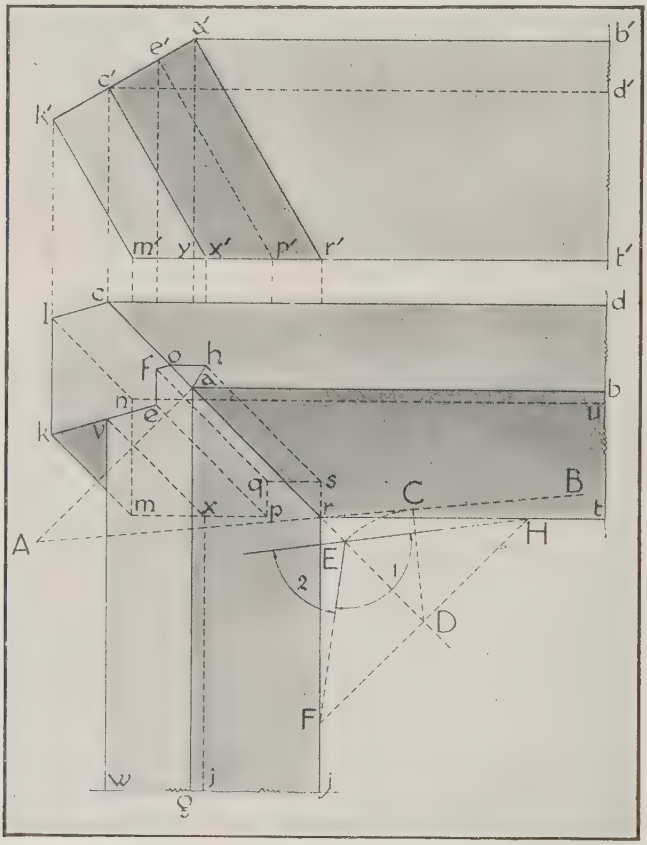


Fig. 160.*

the tongue. Draw the plans of the lower edges, as shown by the full lines *jr*, *rt*, and the dotted lines *un*, *nm*, *mp*, *pq*, *qs*, *sr*, *rt*, and *xi*. *pqrs* is the plan of the lower surface of the tongue. Drop a perpendicular from a^1 to meet a line from *r* in *a*, making the angle $mra = 45°$. Produce *ra* to meet a perpendicular from c^1 in *c*. Draw *cd* and *ab* parallel to *rt* and *ag* parallel to *rj*. From *m* and *n* draw parallels to *ra* to meet a perpendicular from k^1 in *l* and *k*; then *lkmn* is the plan of the sloping

end of the grooved side whose elevation is m^1k^1. Draw a perpendicular from e^1 to meet a parallel to *ra* from *p* in *e*. Join *k*, *e* and *l*, *c*; *ke* and *lc* will be parallel lines, and *ke* produced would pass through *a*. From *e* parallel to *ag* and from *q* parallel to *ra* draw straight lines to intersect in *f*. Draw *fo* parallel to *ke*, *oh* parallel to *ab*, and *sh* parallel to *ra*. Draw *vw* in the same straight line with c^1c; then *vx* is parallel to *ra*, and the plan is completed. It is important to note that most of the points in the plan are determined by the condition that *parallel lines project parallel.*

The bevels for the lines *ve* and *ah* on the upper edges of the sides may be determined as in the last example.

The angle between the sides of the groove *efqp*, *ahrs* and the face of the grooved side is the supplement of the dihedral angle between the sides, and is determined as follows :

Produce *ar* to any point *D* and draw a perpendicular to *ar* through *D* to meet *rj* in *F* and *rt* in *H*. Draw *aA* perpendicular to *ar* and equal to ya^1. From *A* draw a straight line through *r* and produce it to any point *B*. Draw *DC* perpendicular to *rB* and with *D* as centre, *DC* as radius, cut *rD* in *E*. Draw *FE* and *HE*. Then the angle *FEH* (1) is the dihedral angle between the sides and the supplement of *FEH* (2) is the angle between the sides of the groove and the face of the grooved side.

The bevels across the square upper edges may be determined as in the previous example, by turning the edges into horizontal positions.

CHAPTER XI

FURTHER PROBLEMS ON LINES AND PLANES

THE work in this chapter is of a general nature, and although no special applications of the problems are included in the chapter, many important geometrical devices and constructions are introduced. Some of the diagrams appear rather involved, but it should be noted that when the result only is required, and the diagram does not need to be explained, most of the lines may be dispensed with.

Problem 69. *Figs.* 161, 162, *and* 163.—**To place a line of given length and inclination in a given plane.**

In Fig. 161 *vth* is the given inclined plane, θ the given inclination, and a^2b^1 the given length of the line. Draw a^2b^1 equal to the given length, with one extremity in XY and the other in vt and making the given angle with XY. Drop a perpendicular from b^1 to meet XY in b. With b as centre and ba^2 as radius, draw a circular arc to cut ht in a. Join a, b; then ab is the plan and a^1b^1 the elevation of the required line.

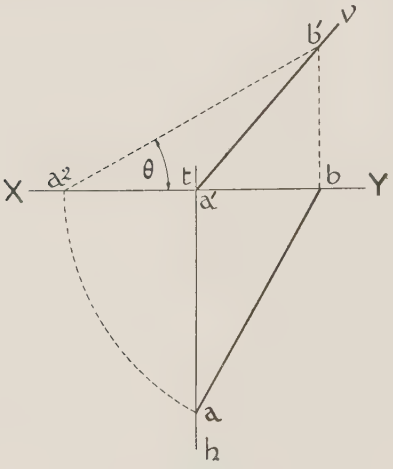


FIG. 161.

For a and a^1 are the projections of a point in the horizontal trace, and b and b^1 the projections of a point in the vertical trace of the plane: therefore the straight line joining A and B is in the plane; and because A is in HP and the height of B is equal to bb^1, and ba equals ba^2, the triangle formed by AB as hypotenuse, the plan ab as base, and the projector bb^1 as perpendicular is equal in every respect to the right-angled triangle a^2b^1b, and the base angle θ is the inclination of AB to HP.

In Fig. 162 the given plane is an acute-angled oblique plane, and in Fig. 163 an obtuse-angled oblique plane. The points in the three

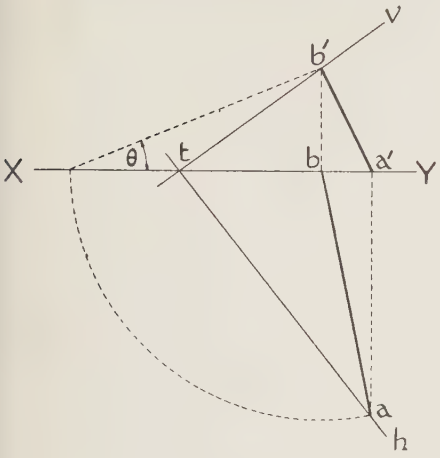


Fig. 162.

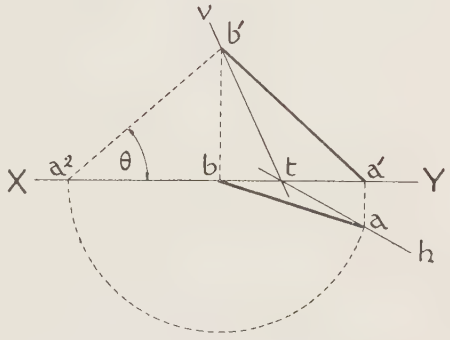


Fig. 163.

figures are lettered to correspond, and since the working is precisely the same in the three cases, no further explanation should be necessary.

Problem 70. *Fig.* 164.—**To determine a plane to bisect a given line perpendicularly.**

ab, a^1b^1 are the projections of the given line.

Bisect *ab* in *c* and draw a perpendicular to *ab* through *c* to meet *XY* in *d*. Draw cc^1 perpendicular to *XY*, and from c^1 draw a parallel to *XY* to meet a perpendicular from *d* in d^1. Draw *vt* through d^1 perpendicular to a^1b^1, and draw *ht* parallel to *cd*. *vth* are the traces of the required plane. For, since the projections of the line are perpendicular to the traces of the plane the line is perpendicular to the plane, and because *cd* and c^1d^1 are the projections of a horizontal line lying in the plane, and *C* is the midpoint of *AB*, the plane bisects the given line perpendicularly as required.

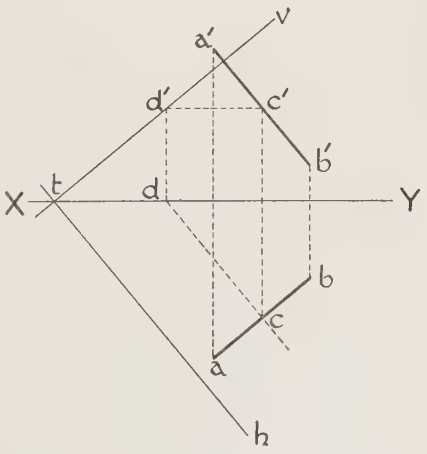


Fig. 164.

Problem 71. *Fig.* 165.—**To place a line of given length in a given plane, the line to make a given angle with one of the traces of the given plane.**

Let *vth* be the traces of the given plane, α the given angle.

At any convenient position draw cc^1 perpendicular to XY and cC perpendicular to ht, C being in a circular arc drawn with t as centre and tc^1 as radius. Draw a straight line from t through C; then htC is the real angle between the traces of the plane. Draw aB equal to the given length of the required line, having a in ht, B in tC (produced if necessary), and the angle taB (α) equal to the given angle. Draw a straight line from B perpendicular to ht to meet XY in b. Project a to a^1 and b to b^1, and draw ab and a^1b^1. Then ab and a^1b^1 are the projections of the required line. For atB is the true shape of the triangle whose plan is abt and whose elevation is a^1b^1t, the triangle having the horizontal trace of the plane between a and t for one side, the vertical trace between b^1 and t for a second side, and the line AB for the third side.

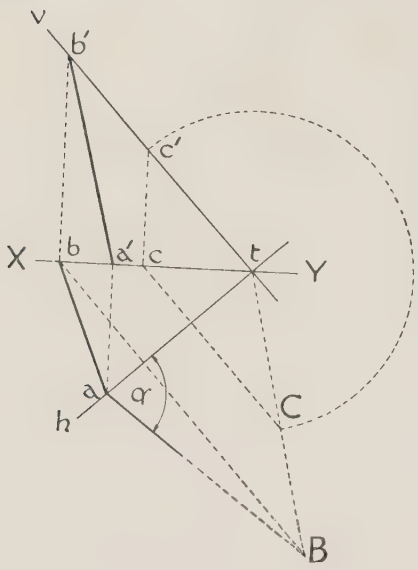


Fig. 165.

Problem 72. *Fig.* 166.—**To determine a plane of given inclination, to contain a given point and be perpendicular to a given plane.**

Let *vth* be the traces of the given plane, p, p^1 the projections of the given point, and θ the given inclination. Draw ap perpendicular to ht and a^1p^1 perpendicular to vt. Produce a^1p^1 to meet XY in c^1, and from c^1 erect a perpendicular to meet ap produced in c.

Draw p^1b^1 to make θ with XY. From p parallel to XY and from b^1 perpendicular to XY draw straight lines to meet in b. With p as centre, pb as radius, draw a circle and draw a tangent to this circle from c. Draw pd parallel to the tangent ch^1 and erect a perpendicular from d to meet a horizontal through p^1 in d^1. Draw a straight line through d^1 to intersect ch^1 on the ground line in t^1. Then d^1t^1 and c^1h^1 are the traces of the required plane.

The solution is based on the following important theorem: *If a line be at right angles to a plane, every plane which contains the line will be at right angles to that plane.*

Then, because ap is at right angles to ht and a^1p^1 is at right angles to vt, ap, a^1p^1 are the projections of a line which is at right angles to the given plane and contains the given point. Also the circle pb is the base circle of a right circular cone, the vertex of the cone being in P, the base in HP, and the base angle equal to θ. Then because the horizontal trace of the plane, $v^1t^1h^1$, contains c, the horizontal trace of the line AP, and is tangent to the base circle of the cone; and because

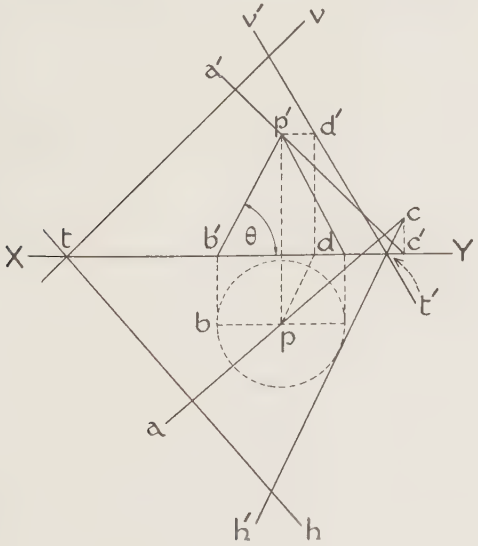


Fig. 166.

also the plane $v^1t^1h^1$ contains the vertex of the cone and, therefore, contains the point P and the line AP, $v^1t^1h^1$ is perpendicular to vth, contains the point P, and is inclined to HP at θ.

Problem 73. *Fig.* 167.—**Given the traces of a plane, *vth*, and the projections of a line *AB* lying in that plane; to determine the traces of a second plane, $v^1t^1h^1$, to contain the given line and make a given angle α with the given plane.**

Since both planes are to contain the given line, the projections of the line will be the projections of the line of intersection of the planes. Determine c and d^1, the horizontal and vertical traces of the given line, and draw dD perpendicular to cd and equal to dd^1; then dcD is the inclination of the line of intersection of the planes. Draw a

perpendicular to *cd* from any point *e* to meet *ht* in *f*, and draw *eg* perpendicular to *cD*. With *e* as centre, *eg* as radius, cut *cd* in *k* and join *k* to *f*. Draw a straight line from *k* to make the given angle α with *kf* and meet *fe* in *m*. Then the horizontal trace of the required plane passes through *c* and *m*, meeting the ground line in t^1; and the vertical trace is through t^1, d^1. The construction should require no detailed explanation if the problems relating to the dihedral angle have been studied.

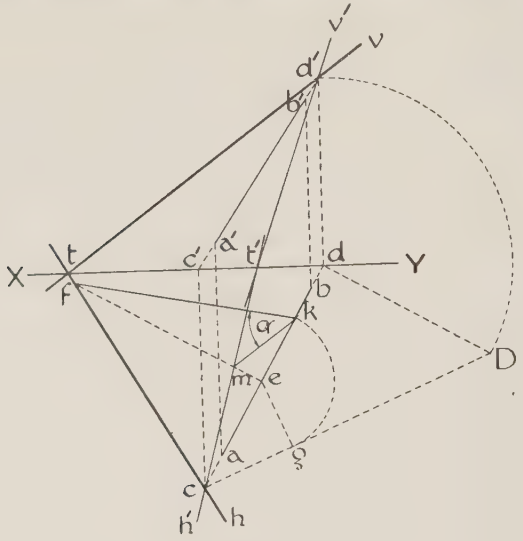


Fig. 167.

Problem 74. *Fig.* 168.—**Given the real angle between the traces of a plane and the inclination of the plane to one of the co-ordinate planes, to determine the relative positions of the ground line and the traces.**

Let α be the given angle between the traces, and θ the inclination of the plane to *HP*.

Draw any two straight lines *at* and *ht* to contain the given angle α. Draw a perpendicular to *ht* to intersect *at* in *a* and *ht* in *b*. Draw *bA* to make the given angle θ with *ab* produced, and make *bA* equal to *ba*. Draw Aa^1 perpendicular to *ab* and draw *XY* through ta^1. Draw a^1a^2 perpendicular to *XY* and equal to a^1A. Draw *vt* through a^2. Then *XY* is the ground line and *vth* the required traces.

For let the triangle *abt* be turned about *tb*; then *a* will move through a circular arc whose plan is aa^1, and when *a* stands immediately above a^1 the inclination of the line *ab* will be equal to θ; and because *ab* and a^1b are both at right angles to *ht*, the plane containing *ab* and having *ht*

for its horizontal trace will be inclined at the given angle θ. Also, because A is in the vertical trace of the plane, its plan a^1 will be in the ground line, and, since bA equals ba, the perpendicular distance of A from XY is equal to a^1A. Or let the triangle ta^2a^1 be turned upwards through 90° about XY, and let the triangle bAa^1 be turned upwards through 90° about ba^1; then the lines a^1A and a^1a^2 will coincide, both being perpendicular to HP from a^1 and of equal length. Then if the triangle tba be turned about tb until ba and bA coincide, ta and ta^2 will also coincide and the plane of the triangle tba will be inclined at the given angle θ and will have ta^2 for its vertical trace.

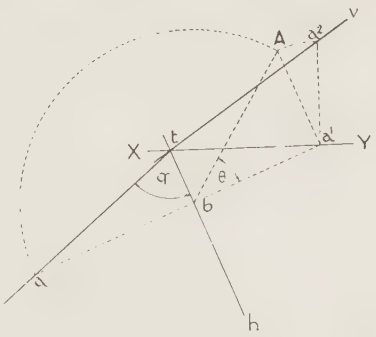


Fig. 168.

Problem 75. *Fig.* 169.—**Given the horizontal traces and the inclinations to *HP* of any two intersecting planes, to determine the plan of the line of intersection.**

Let ht and h^1t^1 be the given traces, α and β the given inclinations.

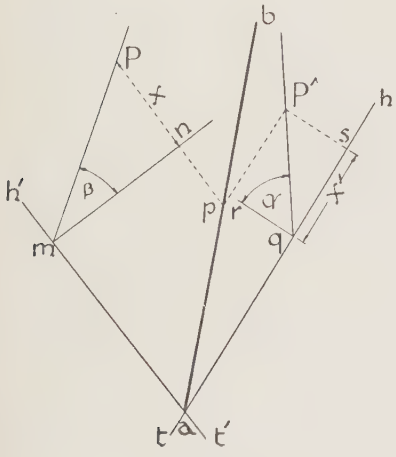


Fig. 169.

Draw any line mn at right angles to h^1t^1 and draw mP to make the given angle β with mn. Draw any parallel Pn to h^1t^1 to obtain x. Draw any line qr at right angles to ht; make the distance qs (x^1) along ht equal to x. Draw qP^1 to make the given angle α with qr, and draw sP^1 parallel to qr. From P^1 draw a parallel to ht to meet Pn produced in p; then a straight line ab through p is the plan of the line of intersection. Pp and P^1p are horizontals, one in each plane, of equal height (x). Then p is a point in the plan of the line of intersection and the point of intersection of the given traces (a) is a second point.

Problem 76. *Fig.* 170.—**Given the horizontal traces of two**

intersecting planes and the inclinations of the planes to any given vertical plane, to determine the projections of the line of intersection.

Let ht and h^1t^1 be the given traces, α and β the given inclinations to the vertical plane on XY.

From a, the point of intersection of the given traces, draw aa^1 perpendicular to XY. Draw ac and ab to make the given inclinations with XY. With a^1 as centre, a^1b and a^1c as radii, draw circular arcs above XY, and draw tangents tv and tv^1 to these arcs from t and t^1. Then vt and v^1t^1 are the vertical traces of the planes. The projections of the line of intersection may now be drawn as already explained. Consider a as the common vertex and aa^1 as the common axis of two right circular cones whose bases are in the vertical plane on XY, the circular arcs, centre a^1, radii a^1b and a^1c, being part elevations of the base circles. Then because each plane has its ht in the vertex, and its vt tangent to the base circle, it is inclined to VP an amount equal to the base angle of the cone.

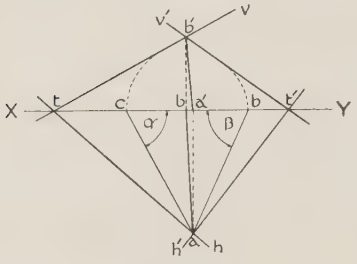


Fig. 170.

Problem 77. *Fig.* 171.—**Given the traces of any two intersecting planes, to determine the traces of a third plane which shall be perpendicular to each of the given planes and contain a given point P.**

Let vth, $v^1t^1h^1$ be the traces of the given planes, and let p and p^1 be the projections of the given point.

First consider the following theorems:

If a straight line be perpendicular to a given plane, any plane which contains that line will be perpendicular to the given plane.

From this it follows that *if two planes which cut each other be each of them perpendicular to a third plane, their line of intersection will also be perpendicular to that plane.*

There are, then, two methods of solution (both shown in the figure): one, by finding a plane to contain the given point and be perpendicular to the line of intersection of the given planes; two, by arranging a perpendicular line to each plane to pass through the given point, and finding a plane to contain the two lines.

In the figure, ab, a^1b^1 are the projections of the line of intersection between the given planes, determined as follows: Produce ht and h^1t^1 to intersect in c and drop a perpendicular to meet XY in c^1; then c and c^1 are the projections of a point in the line of intersection. a and a^1

are also the projections of a point in the line of intersection, and straight lines through c, a, and c^1, a^1 are the plan and elevation of the line of intersection, and any plane perpendicular to this line is perpendicular to both the given planes. It has already been stated that, when a line is perpendicular to a plane, the projections of the line are at right angles to the traces of the plane ; therefore the horizontal trace of the required plane is required to be at right angles to ab, and the vertical trace at right angles to a^1b^1. But the required plane must contain P, and a point may be found in the vertical trace by drawing a perpendicular to ab through p to meet XY in q, and erecting a perpendicular from q to meet a horizontal through p^1 in q^1. Draw v^2t^2 through q^1 perpendicular to c^1b^1, and draw h^2t^2 perpendicular to cb ; then $v^2t^2h^2$ is the required plane.

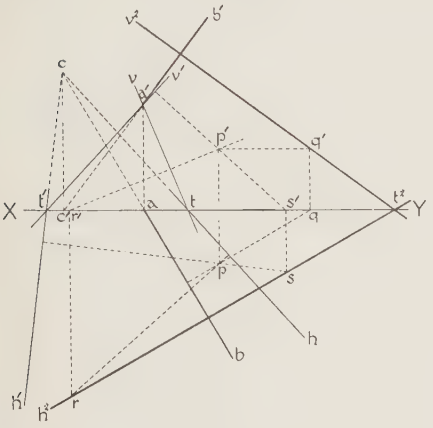


Fig. 171.

Or, draw pr, p^1r^1 perpendicular to ht and vt respectively and draw ps, p^1s^1 perpendicular to h^1t^1 and v^1t^1 respectively. Then h^2t^2 passes through r and s, the horizontal traces of the lines, and v^2t^2 may be found by drawing the projections of the horizontal line through P to obtain q^1, or the vertical traces of the lines PR and PS may be found.

Problem 78. *Fig.* 172.—**Given the traces of a plane and the projections of two points, both situated above the plane, to draw two straight lines, one from each point, to intersect in the plane and make equal angles with it.**

This is equivalent to finding a point in the plane such that the sum of its distances from the given points is a minimum.

Let vth be the traces of the given plane, p, p^1 and q, q^1 the projections of the given points.

Draw q^1a^1 perpendicular to and cutting vt in b^1. From b^1 draw a perpendicular to meet XY in b, and a perpendicular from a^1 to meet ht in a. Join a, b and from q draw a perpendicular to ht to cut ab in g. Draw p^1c^1 to cut vt perpendicularly in d^1 and project d^1 to meet XY in d, and c^1 to meet ht in c. Join c, d and draw a perpendicular to ht from p to meet cd in f. Join g to f. Produce qg to e, making ge equal to gq. Draw pe to cut gf in r. Project g to g^1 and f to f^1 and draw g^1f^1. Project

r to r^1. Draw pr, qr, p^1r^1, and q^1r^1, then PR and QR are the required lines.

For, ab, a^1b^1 are the projections of the line of intersection between the given plane and an inclined plane which contains Q and has aa^1 and a^1b^1 as traces, and because qg and q^1g^1 are perpendicular to ht and vt respectively, the line QG is a perpendicular to the plane vth; and since G is in this line AB, it is the point in which QG meets vth. In the same way f and f^1 may be shown to be the projections of the foot of a perpendicular to the plane vth from P, cd and c^1d^1 being the projections of the line of intersection between vth and an inclined plane containing P. Also, because QG and PF are parallel straight lines, both being perpendicular to vth, they lie in a plane that intersects the given plane vth in the line GF; and because the points P, G, E are collinear, E is also in the plane which contains QG and PF, and the straight line joining P and E intersects the given plane in r. Then, since GE equals QG, and QE is perpendicular to the plane containing GF, the angles EGR, QGR are right angles and the triangle QRE is isosceles, and the angles ERG, QRG are equal; but angle PRF equals angle ERG, these being the opposite angles between two intersecting straight lines; therefore the angles QRG, PRF are equal, and since GR and FR are the projections of QP and PR on the given plane, QR and PR are equally inclined to vth, R being the point in the plane in which they intersect.

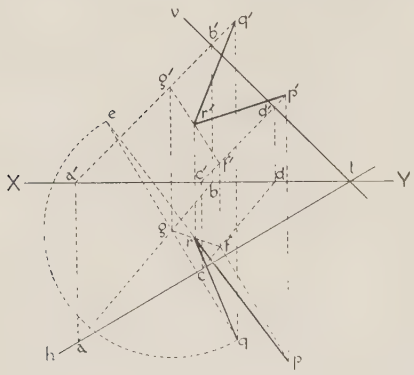


Fig. 172.*

To show that $PR+QR$ is a minimum, consider Fig. 173, in which the points are lettered to correspond with Fig. 172. $QG=GE$, and QE is perpendicular to GF; therefore $QR=ER$, and the angles QRG, ERG, and PRF are equal. Then R is the point in GF such that $QR+PR$ is a minimum. For, take any other point S and join Q to S, P to S, and E to S; because the triangle QES is isosceles, $QS=ES$, and therefore $QS+PS$ equals $ES+PS$. But $PS+ES$ is greater than PE, therefore greater than $PR+QR$, and since S is any point in GF other than R, $PR+QR$ is a minimum.

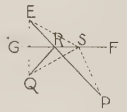


Fig. 173.

PLATE II.

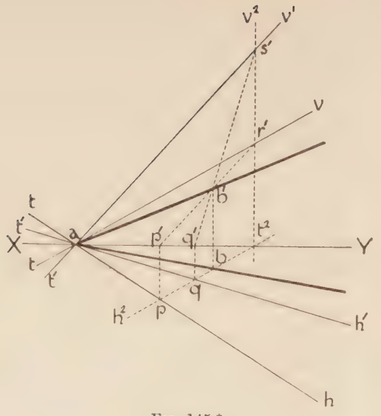


Fig. 145.*

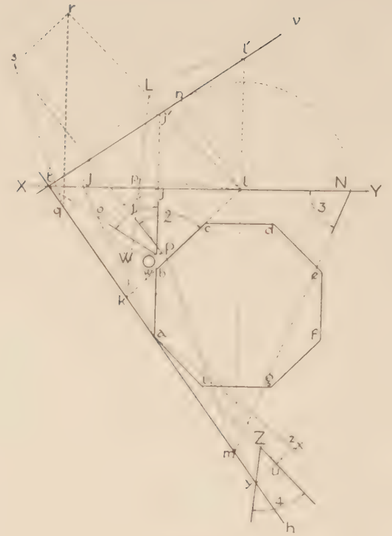


Fig. 151.*

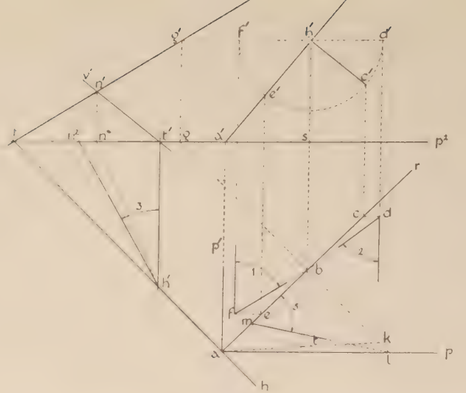


Fig. 156.*

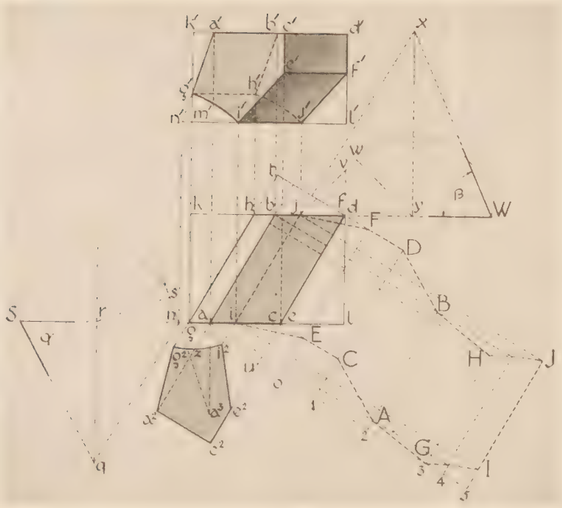


Fig. 158.*

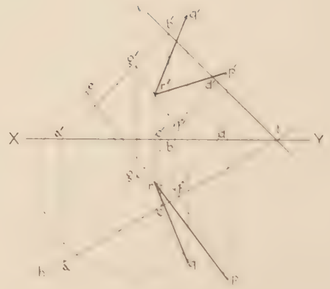


Fig. 172.*

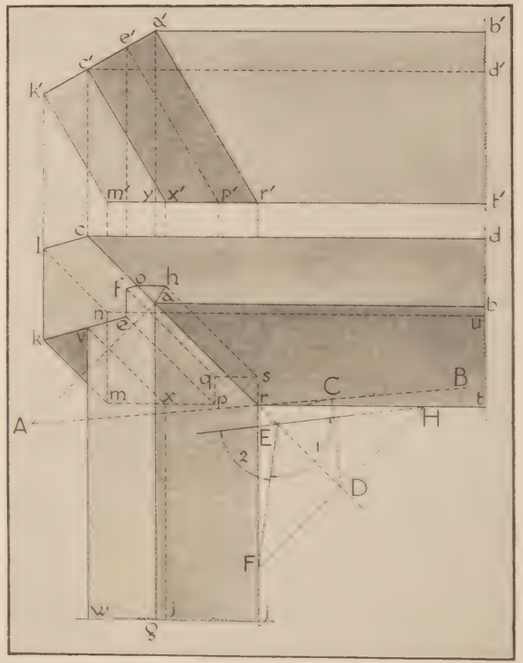


Fig. 160.*

Face page 146.

Problem 79. *Fig.* 174.—**Given the projections of any two non-intersecting straight lines, to draw their common perpendicular; that is, to draw a third straight line to intersect each of the given lines at right angles.**

Let ab, a^1b^1 and cd, c^1d^1 be the projections of the given lines.

The method of solution chosen involves the extremely useful operation of finding a plane to contain one of the given lines and be parallel to the other. When this operation is carried out, this and other more involved problems become greatly simplified.

If a straight line, outside a given plane, be parallel to any straight line in the plane, it will be parallel to the plane.

This theorem is made use of as follows: Draw de parallel to ab and d^1e^1 parallel to a^1b^1, then if e^1 is in XY and c^1c is perpendicular to XY, e is the horizontal trace of a line which intersects one given line CD, and is parallel to the other, AB. Determine f, the horizontal trace of CD; then the straight line through e and f is the ht of a plane containing CD and parallel to AB. Draw X^1Y^1 at right angles to ef and determine a second elevation of the given lines on this ground line. d^2f^2 and b^2g^2 are elevations of the given lines continued to meet HP in f and g, and are parallel. Draw any straight line, as g^2h^2, perpendicular to g^2b^2 and f^2d^2, and project to gh, gh being perpendicular to ef. From h, parallel to ab, draw a line to meet cd in p, and draw pq parallel to gh. Project p to p^1 and q to q^1 and join q^1, p^1. Then pq and p^1q^1 are the projections of the required common perpendicular, and its true length is g^2h^2.

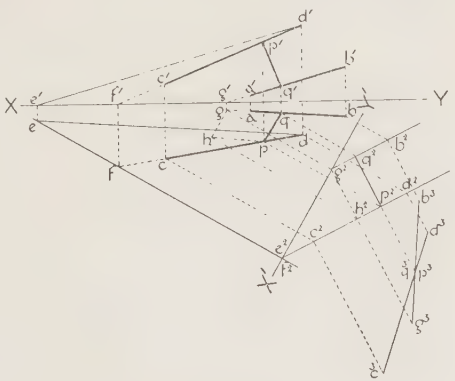


Fig. 174.

For, f^2d^2 is an edge view of the plane that contains CD and is parallel to AB, and since g^2b^2 is an elevation of AB on the same ground line, g^2h^2 is the mean or perpendicular distance between the given lines, and since GH is perpendicular to the plane ef^2d^2, its plan gh is perpendicular to the horizontal trace ef. If the line GH be moved so as to have G always in AB, or AB produced, and to be parallel to its original position, GH, H will move always in the plane containing CD; therefore when GH is at PQ, where p and p^2 are in the same straight line perpendicular

to the ground line, and qq^2 are also in a straight line perpendicular to the ground line, PQ intersects both the given lines at right angles.

If a new plan of the given lines be determined on f^2d^2 as ground line, making the perpendicular distance of each point from f^2d^2 equal to its distance from X^1Y^1 in the original plan (thus, h^2g^3 equals g^2g), the point of intersection, q^3p^3, of the new plan, b^3g^3 and c^3d^3, is an end view of the common perpendicular, and will give the position of the PQ direct.

Problem 80. *Fig.* 175.—**Given the projections of a straight line AB, to determine the traces of a plane of given inclination θ to make a given angle α with the given line.**

Let ab and a^1b^1 be the projections of the given line.

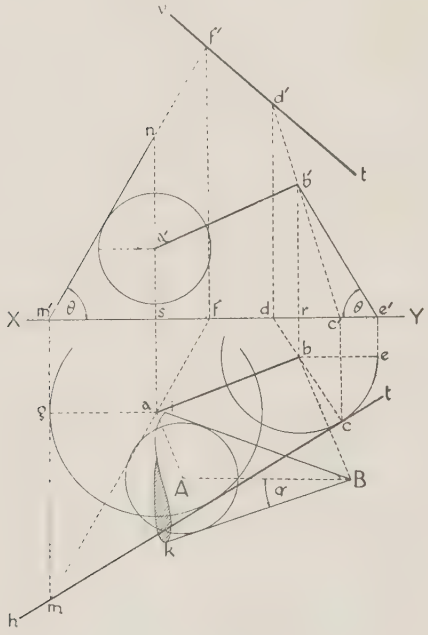


Fig. 175.

Draw aA and bB perpendicular to ab and equal to sa^1 and rb^1 respectively, and join A, B. Then AB is the given line rabatted about its plan into HP.

About AB as axis draw the outline of a right circular cone Bki, whose vertex angle equals twice the given angle α. With A as centre draw a circle to touch Bi and Bk, and with the same radius draw a circle about a^1 as centre. Draw nm^1 to touch the circle, centre a, and make the given angle θ with XY. Draw b^1e^1 to make the same angle θ with XY, and draw circles about a and b as centres to touch perpendiculars from m^1 and e^1 in g and e; then ht, drawn tangent to both these circles, is the horizontal trace of the required plane. The vertical trace may now be determined by any of the methods already shown. In the figure, f^1 and d^1, two points in the vertical trace of the plane, are determined as follows: Produce m^1g to meet ht in m. From m through a draw a straight line to meet XY in f and project f to meet m^1n produced in f^1. From any point c in ht draw a straight line through b to meet XY in d. Project c on to XY at c^1, and from c^1 draw a straight

line through b^1 to meet a projector from d in d^1. Then the required vertical trace passes through f^1 and d^1.

Consider the circle, centre A, as a sphere inscribed in the cone whose axis is AB and whose vertex angle is equal to twice the given angle between the given line and the required plane. Then any plane to touch this cone will make the required angle with the given line. If the cone and sphere are turned about ab so as to bring the axis AB into its real position above its plan ab, the circle, centre a^1, will be the elevation of the inscribed sphere, and because nm^1 is tangent to this elevation circle and is inclined to XY at θ, it represents an outline generator of a second cone enveloping the sphere, this cone having its base in HP and its base angle equal to the given angle θ. Then any plane touching this cone will be inclined to HP at θ, and because the two cones envelop a common sphere, a plane may be found to touch them both. Also, because the required plane must contain the point B, it will touch a third cone whose vertex is in B, whose base angle is θ, and whose base is in HP, and since the circles ag and be are the base circles of the two cones which stand on HP, the horizontal trace of the required plane will touch the two circles. And because f^1 is the vertical trace of a straight line which has one point in ht and a second point in the vertex of one of the upright cones, it is in the vertical trace of the required plane; and similarly d^1 is in the vertical trace; then the plane vt, ht touches the three cones, and is therefore inclined at the given angle θ and makes the given angle α with the given straight line AB.

Problem 81. *Fig.* 176.—**Given the traces of a plane and the projections of a straight line not contained by the plane, to draw the traces of a plane to contain the given line and make a given angle with the given plane.**

Let vth be the given traces, ab and a^1b^1 the given projections, and α the given angle.

Draw X^1Y^1 at right angles to ht and obtain v^2h^2 by drawing ww^3 parallel to ht and making w^2w^3 equal to ww^1. Determine a second elevation b^2 of B on X^1Y^1, and draw b^2c and b^2d to make the given angle α with v^2h^2. Continue b^2c to meet X^1Y^1 in g^1, and produce b^2d to e^1. Bisect g^1e^1 in n^1. Draw b^2l perpendicular to v^2h^2 and draw ml through n^1 parallel to v^2h^2. With l as centre, lm as radius, draw a circular arc to meet a parallel to b^2l from n^1 in o. Draw a perpendicular to ht from b to meet parallels to ht from g^1 and e^1 in g and e. Then ge is the major axis of the elliptical intersection between HP and a right circular cone, vertex in B, base in vth, and base angle α. (The cone is continued below the plane vth to obtain this ellipse.) Draw pq to bisect eg

perpendicularly and make nq and np each equal to n^1o. Then pq, which passes through n^1, is the minor axis of the elliptic section, and the focal points f and f^1 may be found by making pf, pf^1, qf, and qf^1 each equal to ne or ng. Determine s, the horizontal trace of AB, and join s, f. On sf as diameter draw a circle to cut the major circle of the ellipse in r. Draw h^1t^1 through s and r, and draw bz parallel to h^1t^1. Project z to meet a horizontal through b^1 in z^1. Draw v^1t^1 through z^1. Then $v^1t^1h^1$ is the required plane.

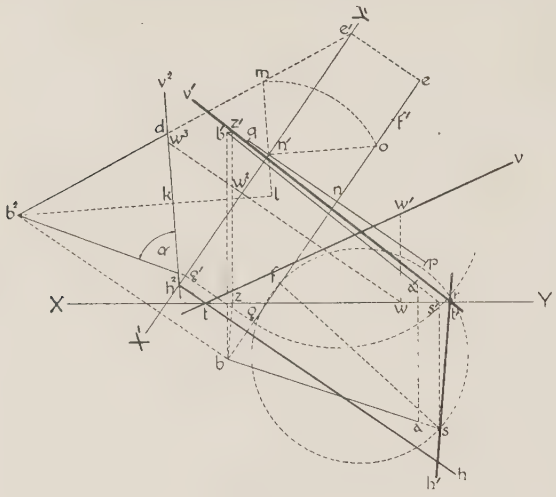


Fig. 176.*

For, the required plane must touch any cone whose vertex is in the given line, its axis at right angles to the given plane, and its base angle equal to the given angle. Then because v^2h^2 is an edge view of the given plane, and b^2 is a projection of B, a point in the line, and b^2c and b^2d make the angle α with v^2h^2, the triangle $b^2a^1e^1$ is the true section of a cone of the required angle, with its vertex in B, made by a vertical plane that contains B and is perpendicular to the given plane. Then, because the axis of the cone is perpendicular to the given plane, it is in the vertical section plane, and the latter plane will contain the major axis of the elliptic section between the cone and HP. (For the major axis of any ellipse lies in the vertical plane which contains the axis of the cone and is perpendicular to the section plane.) And since ge is in the line of intersection between the axial plane and HP, and is equal to g^1e^1, ge is the major axis of the ellipse by HP and the cone. Also, because n^1 is the mid-point of e^1g^1 and lm is perpendicular to the axis bl, the arc mo is part of the circular section of the cone that contains n^1, the elevation of the minor axis; and since the distance through the cone at n^1, perpendicular to the axial plane, is twice n^1o, n^1o equals the semi-minor axis. Then pq is the minor axis and f and f^1 are the foci of the elliptic section. Also, because the plane is re-

quired to contain AB, h^1t^1 will pass through s, and since r is a point in the tangent to the ellipse from s (see chapter on conics), h^1t^1, through s and r, contains the horizontal trace of the line and touches the line of intersection between the cone and HP. And since BZ is a horizontal line in the plane $v^1t^1h^1$, this plane contains the line AB and touches the cone and therefore makes the required angle with the given plane.

Problem 82. *Fig.* 177.—**Given the projections of two non-intersecting straight lines, to draw the projections of a third straight line to intersect and make given angles with the given lines.**

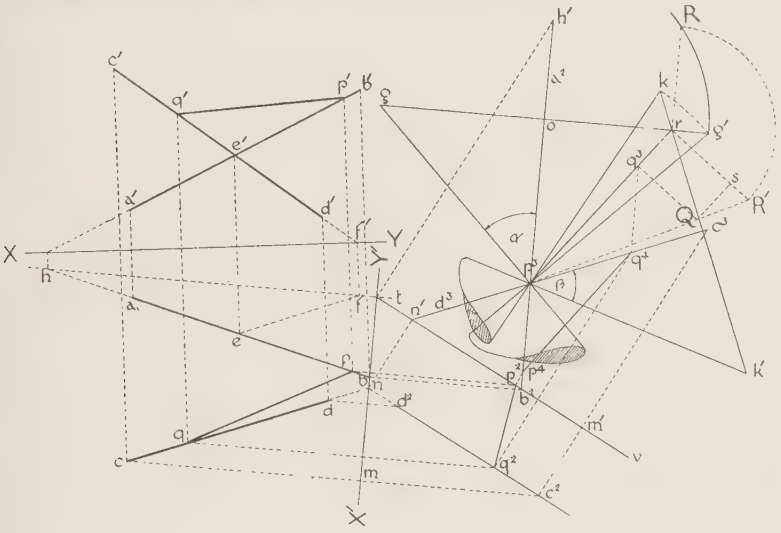


Fig. 177.*

Note.—The solution is possible when the sum of the given angles is greater than the angle between the given lines.

Let ab, a^1b^1 and cd, c^1d^1 be the projections of the given lines, and let the given angles be α, with AB, and β, with CD.

From the point e^1, in which the elevations of the given lines intersect, draw a perpendicular to XY to meet ab in e. Draw ef parallel to cd; then ef and e^1f^1 are the projections of a line which intersects the given line AB and is parallel to the second given line CD, and f is its horizontal trace. Determine h, the horizontal trace of AB, and draw a straight line through h and f. Then hf is the horizontal trace of a plane containing AB and parallel to CD. Draw X^1Y^1 through b, at right angles to hf, and project c to c^2, and b to b^2. Produce cd to n and draw nc^2. Draw vt through b^2; then nc^2 and vt are parallel, since vt is an

edge view of the plane containing AB and parallel to CD. Draw th^1 perpendicular to vt and equal to th, and join h^1b^2. Draw nn^1 and c^2c^3 perpendicular to vt and make m^1c^3 equal to mc. Join n^1, c^3. Then h^1b^2, n^1c^3 are projections of the given lines, produced on to the plane vth, and since vth is parallel to both lines, the angle between these projections is the real angle between the given lines. Through the point p^3, in which h^1b^2 and n^1c^3 intersect, draw p^3g and p^3g^1, each to make the given angle α with h^1b^2, and draw p^3k and p^3k^1, each to make the second given angle β with n^1c^3. Make p^3g^1 equal to p^3k and draw g^1g at right angles to h^1b^2, and kk^1 at right angles to n^1c^3. From the point r, in which gg^1 intersects kk^1, draw a perpendicular to gg^1 to meet a circular arc, drawn with o as centre, og^1 as radius, in R. Join r to p^3 and draw rR^1 perpendicular to rp^3 and equal to rR. Join R^1p^3. From r set off the distance rs on rR equal to nn^1, the mean distance between the given lines, and draw a parallel to rp^3 from s to cut R^1p^3 in Q. Draw a parallel to rs from Q to meet rp^3 in q^3. From q^3 draw a parallel to h^1b^2 to meet n^1c^3 in q^4, and draw q^4p^4 parallel to rp^3, then q^4p^4 is a projection of the required line. Project q^4 to q^2, q^2 to q, and q to q^1. Project p^4 to p^2, p^2 to p, and p to p^1. Draw qp, q^1p^1, and q^2p^2. Then qp and q^1p^1 are the projections of the required line. For since h^1b^2 and n^1c^3 are projections of the given lines on to a plane parallel to both lines, p^3gg^1 and p^3kk^1 are the projections of two right circular cones, each having its axis in one of the given lines and its generators inclined at one of the given angles to the line containing the axis. Since p^3k equals p^3g^1, all the generators *in both the cones* are of equal length; and because gg^1 and kk^1 intersect in r, rp^3 *is a generator of both cones and makes the required angles with the given lines.* Because g^1R is part of the base circle of one of the cones, and rR^1 equals rR, the angle rp^3R^1 equals the inclination of the common generator rp^3 to the plane containing AB and parallel to CD; and since q^3Q is equal to the distance between CD and the plane containing AB, p^3Q is the true length of that part of the generator rp^3 that may have an extremity in each of the given lines and remain parallel to its original direction. Now suppose the cone gg^1 to move without turning, with its axis always in AB until q^3 is at q^4, and p^3 is at p^4; then because P is in the axis of the cone, it is in AB and in the plane containing AB, and because q^3Q equals the perpendicular distance between this plane and the line CD, when Q is at q^4 it is in CD, and since p^3p^4 is parallel to p^3q^3, PQ is parallel to its original direction and therefore intersects and makes the required angles with the given lines.

A second solution may be obtained by sliding PQ in the direction parallel to CD.

Problem 83. *Fig.* 178.—**To draw the traces of the two planes that will contain a given line AB and be at a given distance from a given point P.**

Let ab, a^1b^1 be the projections of the given line, p, p^1 the projections of the given point.

Determine c and d^1, the horizontal and the vertical traces of AB. About p and p^1 as centres draw the projections of a sphere whose centre is P and whose radius equals the distance of the required planes from the given point. Then each of the required planes will touch the sphere, centre P, and have its horizontal trace through c and its

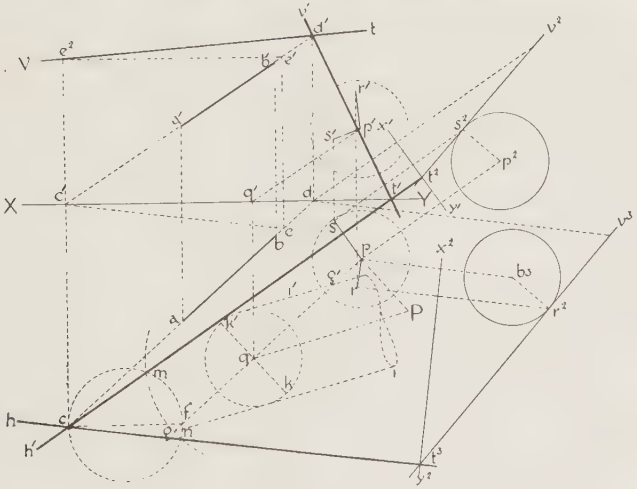


Fig. 178.*

vertical trace through d^1. But any plane that contains the line AB and touches the sphere, centre P, will touch a cylinder whose axis is through P parallel to AB and whose diameter equals the diameter of the sphere.

Draw pq parallel to ab and p^1q^1 parallel to a^1b^1; then PQ is the axis of the cylinder and q is its horizontal trace. Draw pP at right angles to pq and equal to the height of P, and join Pq; then Pq is the axis of the cylinder rabatted into HP. About q as centre draw a circle equal to the diameter of the sphere and draw tangents to this circle parallel to Pq to cut pq in g^1 and pq produced in g. Draw the diameter kk^1 perpendicular to gg^1; then gg^1 is the major axis, kk^1 the minor axis, of the elliptic line of intersection between the cylinder and HP, and tangents to the ellipse from c are the horizontal traces of the required planes. The tangents may be determined without

drawing the curve, as follows: Determine the focus f and draw a circle on cf as diameter to cut the major circle of the ellipse in n and m; then straight lines through c, n and c, m are tangents and are the required horizontal traces.

NOTE.—When the use of the focus nearer to c results in giving the distances cm, cn too small to ensure reasonable accuracy, the second focus may be employed, in precisely the same manner, to give points in the tangents from c.

The vertical trace v^1t^1 may now be drawn through d^1, and a second point e^2 in vt may be found by drawing any line, as ec^1, parallel to hn through any point e in ab, and finding e^2, the vertical trace of the line.

An edge view of each plane, with a projection of P, is shown in the figure, and the projections of perpendiculars to the planes from P are shown at rp, r^1p^1; sp, s^1p^1.

Because the horizontal trace of the given line is in the horizontal traces of both planes and the vertical trace of the line is in the vertical traces of both planes, each plane contains the given line AB. Also, because the axis of the cylinder is parallel to AB, every generator of the cylinder is parallel to AB. And since the horizontal trace of each plane is tangent to the elliptic section of the cylinder, each plane contains one generator of the cylinder; for *if one of two parallel lines be contained by a plane, the second line is wholly within or wholly without that plane.* Then, because the cylinder envelops the sphere whose centre is in the given point and whose radius is equal to the required distance between each plane and the given point, each plane touches the sphere, and therefore contains the given line AB and is at the given distance from the given point P.

Problem 84. *Fig.* 179.—**To determine a plane to intersect a given line in a given point, to make a given angle with the given line, and to be at a given distance from a second given point.**

Let ab, a^1b^1 be the projections of the given line and p, p^1 the projections of the given point.

Let the plane be required to intersect AB in B, make the angle α with AB, and be at a distance z from the point P.

Draw ba^2 parallel to XY and equal to ba.

Draw a perpendicular from a^2 to meet a horizontal from a^1 in a^3, and join a^3 to b^1.

Draw b^1m to make the given angle α with b^1a^3, and draw a circle of radius z to touch mb^1 and have its centre o^3 in a^3b^1. Draw a horizontal from o^3 to meet a^1b^1 in o^1, and draw a circle of radius z about o^1 as centre. Determine o, the plan of the point O, whose elevation is o^1, and about o, p, and p^1 as centres draw circles of radius z. Now

suppose the two circles, centres p, p^1, and the two circles, centres o and o^1, to be the projections of two spheres; one sphere centred in the given point P, the second having its centre in AB and inscribed in a cone,

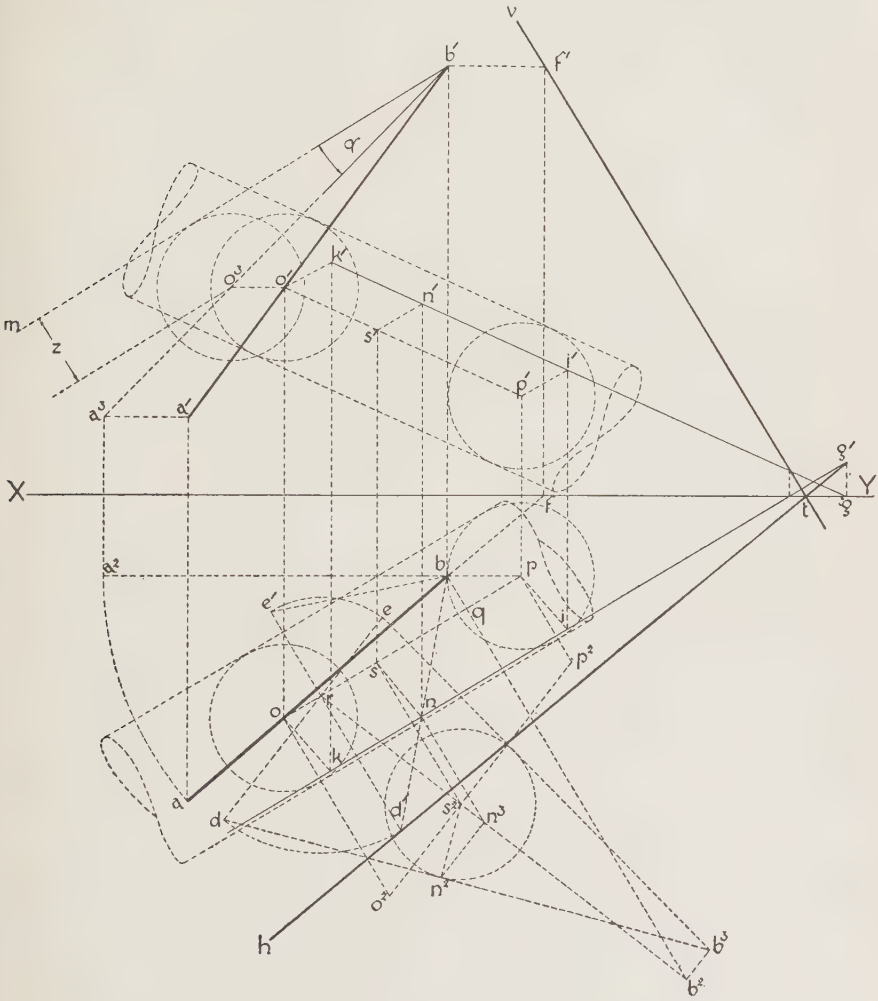


Fig. 179.*

vertex B, semi-vertical angle equal to α. Then if a cylinder be made to envelop the two spheres, the problem is reduced to that of finding a plane to contain the point B and to touch the cylinder.

Draw op and o^1p^1, and on op as ground line determine auxiliary

elevations of O, B, and P, as shown at o^2, p^2, and b^2. Draw b^2s^2 perpendicular to the line joining o^2p^2, and produce it to meet op in r. Draw b^2b^3 perpendicular to b^2r and equal to the distance of b from op. About s^2 as centre draw a circle of radius z and draw tangents to this circle from b^3 to meet a perpendicular to b^2s^2 through r in d and e. Draw a perpendicular to op through r, and with r as centre turn e and d to cut it in e^1 and d^1. Then d^1 is a point in the horizontal trace of one plane, e^1 in the horizontal trace of a second plane that fulfils the required conditions. Project n^2 to meet b^2s^2 perpendicularly in n^3, join b, d^1, and draw a perpendicular to op from n^3 to meet bd^1 in n. Then n is the plan of a point in the line of contact between the cylinder and the required plane. Draw nn^1 perpendicular to XY and make the height of n^1 above XY equal to the distance of n^3 from op. Then n^1 is the elevation of N. Determine g^1, the horizontal trace of the generator of the cylinder containing N. Then ht passes through d^1 and g^1, and since the plane is required to contain B, the point f^1 in vt may be found in the ordinary way.

The projections of the points of contact between the plane and the two spheres, centres O and P, are k, k^1 and i, i^1, therefore OK and PI are perpendiculars to the plane vth.

First consider the two lines b^1m and b^1a^3 containing the given angle a. Because b^1a^3 is parallel to VP, b^1m may represent an outline generator of a cone having its axis in AB and its generators inclined to the axis at the given angle a. Now consider the circle, centre o^3, radius equal to the given distance z, as a sphere inscribed in this cone. Then, when the line AB is brought back into the given position shown at ab, a^1b^1, the centre of the sphere is at o^1, and any plane to touch the sphere and contain B will touch the cone and therefore make the required angle with AB. But the required plane must also touch a second sphere, of radius z, having its centre in the given point P, and the circles, centres p and p^1, are projections of this sphere. Since the two spheres are of equal radius, a cylinder may be introduced to envelop them, as shown in projection by dotted lines, and any plane to pass through B and touch the cylinder will touch both spheres and therefore touch the cone on AB as axis. Now consider the line o^2p^2. Because oo^2 is equal to the height of O and pp^2 is equal to the height of P, o^2p^2 is the axis of the cylinder rabatted about its plan, op, into HP. And since b^2q is at right angles to op and equal to the height of B, b^2 is a rabatment *of the projection of B on to a vertical plane containing the axis,* this rabatment being also about op.

Because b^2r is at right angles to o^2p^2, it is the rabatment of a line which intersects the axis of the cylinder perpendicularly,

and contains the projection of B on the vertical plane containing the axis. Now let the circle, centre s^2, radius z, represent a sphere with its centre in the point of intersection between the axis of the cylinder and the line b^2r perpendicular to the axis. Then, if the cylinder be cut by a plane through B perpendicular to the axis and lines be drawn from B to HP on this section plane tangent to the section of the cylinder, the required plane must contain these lines and the triangle formed by the lines and the line joining their horizontal traces will be equal to the triangle b^3de, where b^2b^3 is equal to bq and at right angles to b^2r, and d and e are the horizontal traces of the tangents from B. Actually, however, b^3de is a rabatment of the section plane through 90° about b^2r, de being at right angles to b^2r. If, then, d^1r and e^1r are drawn perpendicular to op and equal to dr and er respectively, d^1 and e^1 are the actual positions of the horizontal traces of the tangent lines from B on the section plane, and be^1d^1 is the plan of the triangle b^3de.

Consider only the plane through d^1; because n^2 is the point of contact between the circle, centre s^2, and the line b^3d, it is the rabatment of a point of contact between the inscribed sphere and the line Bd, and because n^2 turns first to n^3 and then to n in bd^1, the line of contact between the required plane and the cylinder passes through n in plan and n^1 in elevation, and its traces are in the traces of the required plane.

NOTE.—There are four solutions possible in the foregoing problem, but only two may be obtained when the cylinder is introduced; for the two spheres enveloped by the cylinder must be on the same side of the plane. When the required plane is required to pass between the given point and the given line a different method of solution must be used, as shown in Fig. 180, and explained in the following problem:

Problem 85. *Fig.* 180.—**To determine a plane to make a given angle α with a given line AB, intersect the given line in a given point C, and be at a given distance z from a given point P; the plane to pass between A and P.**

Let ab and a^1b^1 be the projections of the given line, and p, p^1 be the projections of the given point.

Determine d^1, the vertical trace of the given line, and about its plan, d, turn a to A. Join A to d^1. Let c and c^1 be the projections of the point in which the required plane shall intersect AB. Project c^1 horizontally to C and draw Cu to make the given angle α with CA. Draw a parallel to Cu at the given distance z to meet Ad^1 in E, and draw any circle to touch this parallel and have its centre, O, in Ad^1. From E draw a second tangent to the circle, centre O, to meet XY in G. Project O and E horizontally to meet a^1d^1 in o^1 and e^1 respectively. Produce Cc^1 to meet EG in s, and the parallel to Cu in q, and bisect sq

in r. Draw a straight line through E and r to meet XY in K. Draw a perpendicular to XY from O to meet the circumference of the circle, centre O, in I, and through E and I draw a straight line to meet XY in F. About d as centre, turn K, F, and G to k, f, and g. Draw mj through e and p, and m^1j through e^1 and p^1, mm^1 and jj^1 being perpendiculars to XY. Join j to f. With k as centre, kg as radius, draw a circular arc to cut a circle on fj as diameter in i. Draw a straight line through i and j to meet XY in n, and join n, m^1. At any convenient position draw xy perpendicular to ni, and determine p^2, the auxiliary

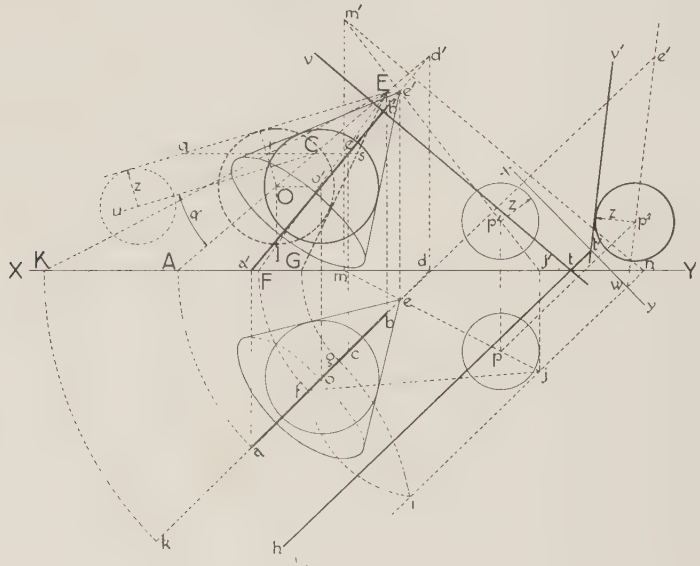


Fig. 180.*

elevation of P on xy as ground line. From w, in which xy cuts ni, draw a straight line through p^2. With the given distance z as radius draw a circle about p^2, and tangent to this circle and parallel to p^2e^2 draw v^1t^1 ; t^1 being in xy. Draw ht^1 parallel to ni to cut XY in t, and draw vt parallel to m^1n. Then vth is a plane satisfying the conditions.

For Ad^1 is the given line rabatted into VP, and EG and Eq are outline generators of a right circular cone having its axis in the given line, its vertex angle equal to the required angle between the line and the plane, and each of its generators the given distance z from C. Let the circle, centre O, represent a sphere inscribed in the cone ; then, when AB is brought back into the given position ab, a^1b^1, O has moved to o^1 in elevation and o in plan, and the *full* circles, centres o and o^1, are the projections of the sphere. And since E has moved to e^1, tangents

to the projections of the sphere from e, e^1 give the projections of the cone about the given line. The problem is now to find a plane to touch the cone and contain the second given point P; for since the surface of the cone is the given distance z from C, the required plane will be parallel to this plane and at the given distance, z, from it.

First, then, to determine the plane to touch the cone and contain P. Since this plane must contain P and the vertex of the cone, its traces will pass through the traces of the straight line cp, e^1p^1. Then j is a point in the horizontal trace and m^1 is a point in the vertical trace of the plane. And since the plane must touch the cone, its horizontal trace will be a tangent to the elliptic intersection between the cone and HP. Then, because qs is parallel to XY and r is its mid-point, and Er produced meets XY in K, K is the centre of the elliptic intersection when the axis of the cone is in VP, and k is its true position. Also KG is the semi-major axis of the ellipse, and kq equals KG. Now imagine the circle, centre O, to move with its centre in Ad^1, increasing in diameter so as to be always tangent to EG and Eq until it touches XY; then F, in EI produced, will be the point of contact, and regarding the circle as a focal sphere, F is the focus of the ellipse in rabatment, f its true position. And since gi is part of the major circle of the ellipse and fi part of a circle on jf as diameter, ji is a tangent to the elliptical intersection between the cone and HP and is the horizontal trace of the plane to touch the cone and pass through P, and mn is the vertical trace. If a sphere of radius z be now placed with its centre in P, a parallel plane to touch this sphere on the side nearer the cone is the one required.

The figure should make the remaining construction clear. A second plane may be determined by drawing the second tangent to the ellipse from j; this would pass through the point of intersection between the circular arcs through g and f on the other side of ab.

CHAPTER XII

THE SPHERE

A SPHERE may be said to be generated by a semicircle making a complete revolution about its diameter, which remains fixed.

Every plane section of a sphere is a circle; the sections through the centre of the sphere are called great circles, and are each equal to the generating circle; all other sections are called small circles.

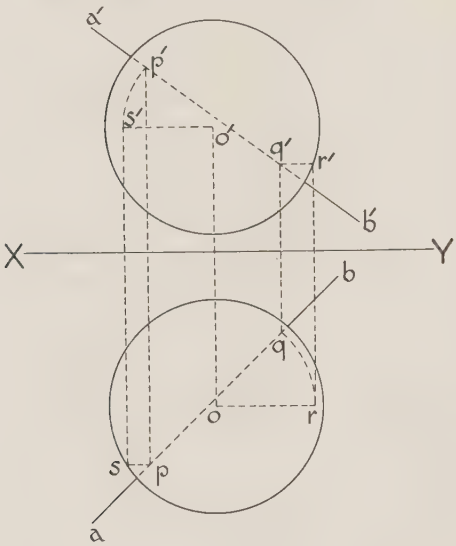


FIG. 181.

All points on the surface of a sphere are equidistant from the centre, the distance being the radius of the sphere.

The orthographic projection of a sphere is a circle equal to a great circle of that sphere.

Problem 86. *Fig.* 181. —**Given one projection of a point on the surface of a sphere, to determine the second projection.**

Let the circles, centres o and o^1, be the projections of a sphere and let p be the given projection of a point on the surface of the sphere, above the centre.

Draw a straight line from p parallel to XY to meet the plan circle in s. Project s to s^1, o^1s^1 being parallel to XY. With o^1 as centre, o^1s^1 as radius, draw a circular arc to meet a projector from p in p^1. Then p and p^1 are the projections of a point P on the surface of the sphere. Or, let q be the plan of a point on the surface of the lower half of the sphere. Draw or parallel to XY, and turn q about o to r. Project r

PLATE III.

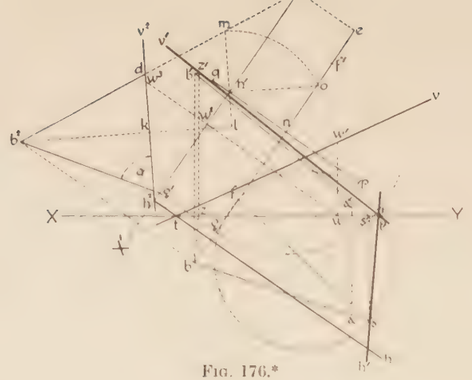


Fig. 176.*

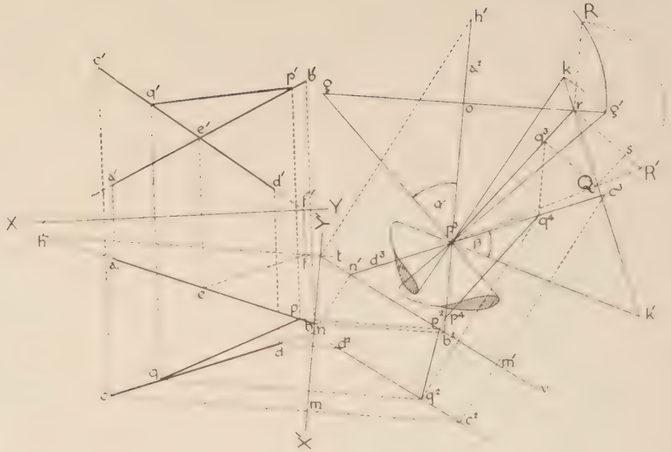


Fig. 177.*

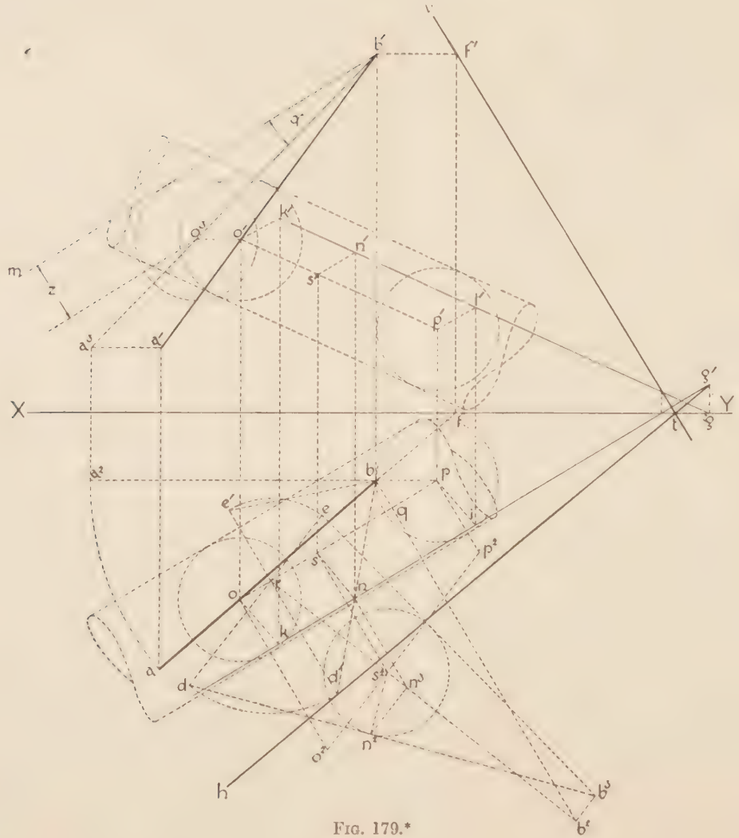


Fig. 179.*

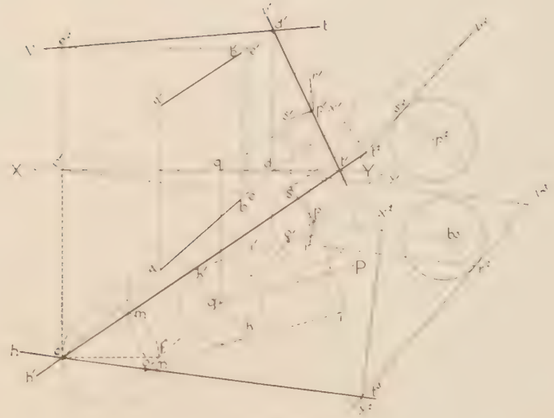


Fig. 178.*

to r^1 and draw a projector from q to meet a horizontal from r^1 in q^1. Then q^1 is the elevation corresponding to the plan q. The projections of a straight line which intersects the surface in the points P and Q are shown at ab and a^1b^1.

NOTE.—The arc s^1p^1 is the elevation of part of the circular section of the sphere by a vertical plane through P parallel to VP, sp being part plan of the section, and arc qr is the plan of part of the horizontal section through Q, q^1r^1 being part of the elevation of this section.

Problem 87. *Fig.* 182. -**Given the projections of a straight line and the projections of a sphere, to determine the points in which the line intersects the spherical surface.**

Let ab and a^1b^1 be the projections of the given line, and o, o^1 the projections of the centre of the sphere. Draw o^1r perpendicular to a^1b^1. With o^1 as centre, o^1r as radius, cut the perpendicular through o^1 in s. Through s and parallel to XY draw e^1f^1. From c^1 and d^1, the points in which a^1b^1 cuts the elevation circle, drop perpendiculars to meet ab in c and d. From c and d draw parallels to XY to meet projectors from e^1 and f^1 in e and f. Draw a straight line through o to cut ee^1 in g and f^1f, produced, in h. About o as centre, turn g to n and h to m, and project n and m horizontally to meet ab in p and q. Project p to p^1 and q to q^1. Then pp^1 and qq^1 are the projections of the required points.

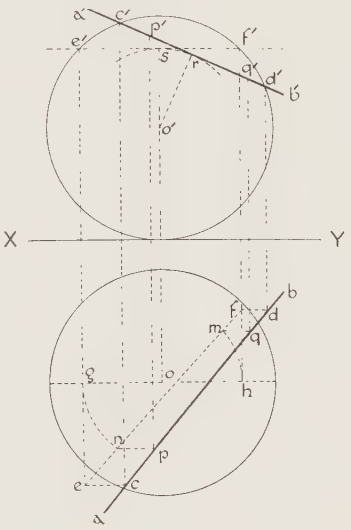


FIG. 182.

For, let a plane containing AB and perpendicular to VP move about the centre of the sphere into the horizontal position through s. Then the points C and D in the line AB have moved to the position shown by e, e^1 and f, f^1, and the plan of the horizontal section of the sphere contains the points g, n, h, and m. And because m and n are the plans of points in the horizontal position of AB, they are the horizontal positions of the points of intersection, and when the plane and line move back into their original positions, n moves to p and m moves to q.

Problem 88. *Fig.* 183.—**To determine a plane to touch a given sphere in a given point.**

Let o and o^1 be the projections of the centre of the given sphere, and p and p^1 the projections of the given point. Join o, p and o^1p^1. Then, if the plane is to touch the sphere at P, it must be perpendicular to the radius OP.

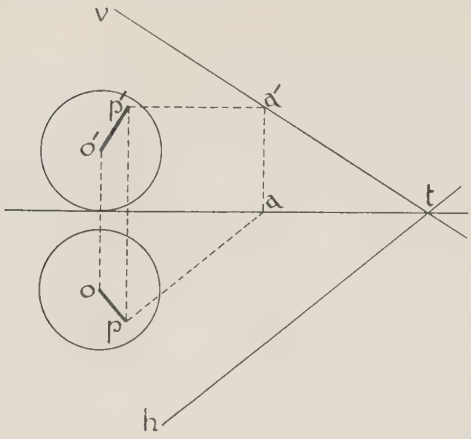


Fig. 183.

The construction should require no explanation.

Problem 89. *Fig.* 184.—**To determine the elevation of the lines of intersection between a sphere and vertical planes through the centre.**

Let o and o^1 be the projections of a hemisphere resting with its circular face in HP, oj and oe the horizontal traces of two vertical

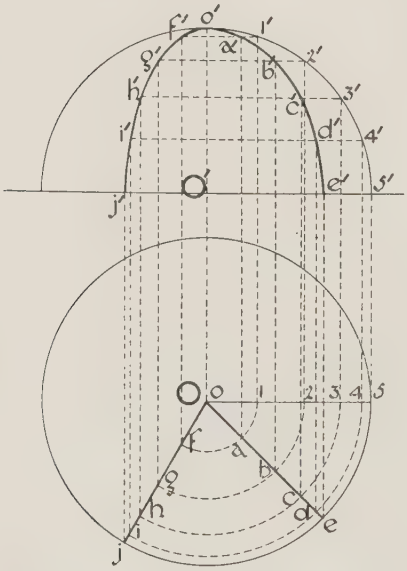


Fig. 184

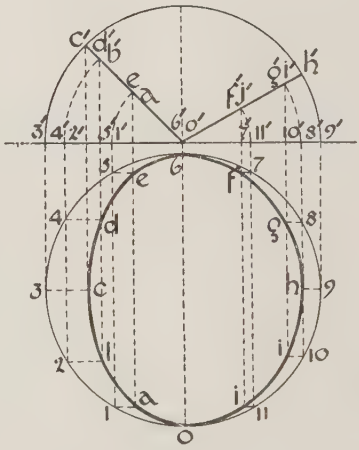


Fig. 185.

cutting planes. Draw the projections of any number of horizontal sections, as $fa1$, $gb2$, to obtain a^1b^1 . . . e^1 and f^1g^1 . . . j^1. Or, since

the elevation curves are elliptical, they may be trammelled about the semiaxes O^1o^1, O^1e^1, and O^1o^1, O^1j^1.

Problem 90. *Fig.* 185.—**To draw the plans of the lines of intersection between a sphere and inclined planes through the centre.**

The diagram should explain the construction.

Problem 91. *Figs.* 186 *and* 187.—**To determine the section of a sphere by an oblique cutting plane.**

First method, Fig. 186. Let o, o^1 be the projections of the centre of the sphere and *vth* the traces of the cutting plane. Determine an edge view of the plane as shown by v^1t on xy, and draw a new elevation of the part of the sphere that is below the plane, as at centre o^2, to

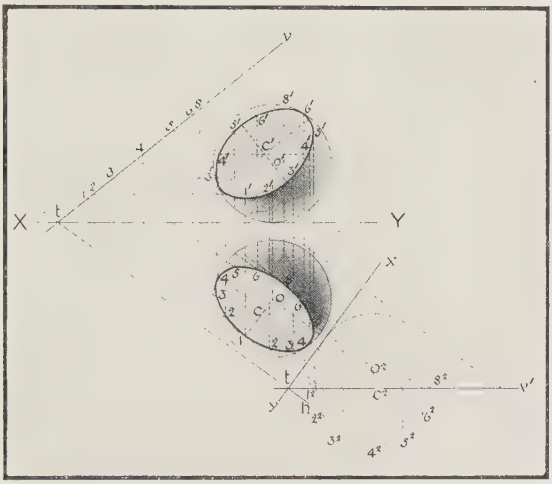


Fig. 186.

meet v^1t in points 1^2 and 8^2. Draw o^2c^2 perpendicular to v^1t, and with c^2 as centre draw the semicircle $1^2 2^2$ . . . 8^2. Divide this semicircle into any number of parts, and drop perpendiculars from the division points on to v^1t. From the feet of these perpendiculars draw parallels to *ht* to represent the plans of horizontal lines in the plane, and determine their elevations. Draw 1-8 through o perpendicular to *ht*, and set off on each side of 1-8 along the horizontals the lengths of the appropriate perpendiculars from the semicircle to v^1t; thus, the two distances $c4$ are each equal to $c^2 4^2$.

Note.—The semicircle $1^2 2^2$ . . . 8^2 is half the true section of the sphere by the plane, and the perpendiculars $c^2 4^2$, etc., etc., are horizontal lines in the plane.

The elevation of the section is found by direct projection on to the

elevations of the horizontals. The points c and c^1 are the centres of the elliptic projections of the section, and the principal axes are through c and c^1, parallel and perpendicular to the traces of the plane.

Second method, Fig. 187.—Through o, the plan of the centre of the sphere, draw a straight line to meet ht perpendicularly in a. Then ao is the direction of the minor axis of the plan ellipse.

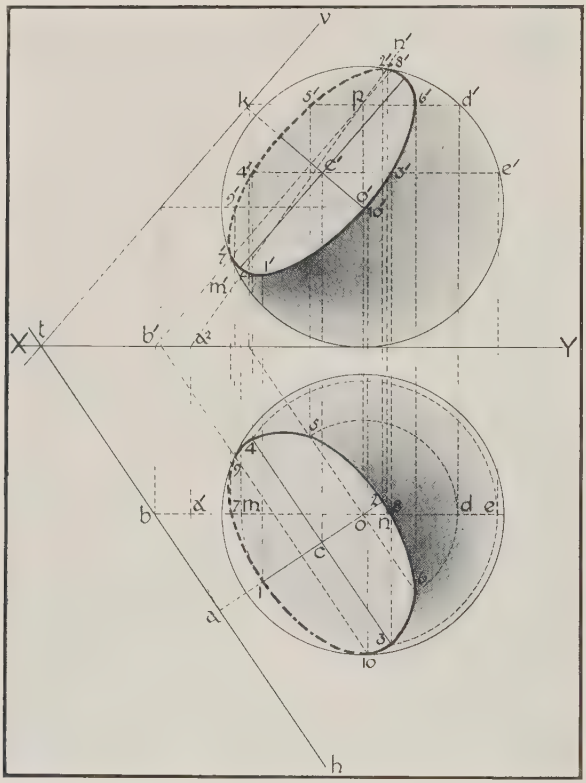


Fig. 187.

Draw ob parallel to XY, and with o as centre turn a to a^1. From a^1 draw a perpendicular to meet XY in a^2. Determine p, the elevation of the point in the plane directly above O, and draw a straight line through a^2p to cut the elevation circle in points m^1 and n^1. Project m^1 and n^1 to m and n respectively, and turn m and n about o to 1 and 2. Then 1-2 is the minor axis of the plan ellipse.

Bisect 1-2 in c; then the major axis of the plan ellipse is parallel to ht through c. Draw o^1k at right angles to vt and project c to c^1;

then c^1 is the centre of the elevation ellipse. Draw a horizontal through c^1 to meet the elevation circle in e^1 and project e^1 to meet *bo* produced in *e*. With *o* as centre, *oe* as radius, draw a circle to meet the major axis of the plan ellipse in points 3 and 4; then 3 and 4 are the extremities of the axis, and the plan ellipse may now be trammelled if desired. The elevations of the points 1, 2, 3, and 4 may be found by projection, $m^1 1^1$ and $n^1 2^1$ being horizontal lines. The extremities of the elevation axes may be found precisely as in plan, or any number of points found in the two ellipses, as follows: From any point in the elevation circle, as d^1, draw a horizontal line in the plane and find its plan. Project d^1 to *d*, in *oe*, and with *o* as centre, *od* as radius, draw a circle to cut the plan of the horizontal in 5 and 6, and project to 5^1 and 6^1.

To determine the points in which the curves pass behind and below the solid. The part of the plan ellipse lying between *XY* and the line *be* is behind the elevation outline circle and would not be seen in elevation, except as a dotted line, unless the portion of the sphere above the plane were removed. The points 7 and 8 which lie in the elevation circle may be found by projecting *b* and n^1 and drawing a parallel to *vt* from b^1 to cut the elevation circle in 7^1 and 8^1.

NOTE.—A parallel to *vt* from b^1 passes through *p*; therefore, when *p* is determined, b^1 is unnecessary. Also, the elevation ellipse and outline circle meet tangentially in 7^1 and 8^1, and the elliptic curve between these points on the side nearer *vt* is dotted.

The points 9 and 10 at which the curve passes below the plan circle are found in a similar manner, the elevation of the horizontal line containing these points being through o^1.

Problem 92. *Fig.* 188.—**Given the projections of four points *A*, *B*, *C*, and *D* on the surface of a sphere, to determine the centre and radius of the sphere.**

Let aa^1, bb^1, cc^1, and dd^1 be the projections of the given points.

To understand the nature of the problem, it is necessary to consider the following properties of the sphere and of the planes and lines involved:

Given any *three* points, not in the same straight line, any number of spherical surfaces may be found to contain them. For, since any three points will lie in one plane, the three straight lines joining the points also lie in one plane; and the three planes which bisect these lines perpendicularly will intersect in a straight line, which will be perpendicular to the plane containing the lines and will be the locus of a point which is always equidistant from the three given points; therefore any point in the line of intersection of the three planes may be the centre of a sphere having the three given points in its surface.

The smallest sphere to contain the three points will have its centre in the point of intersection between the plane containing the points and the line of intersection of the planes bisecting the straight lines between the points.

Given any *four* points, no *three* of which are in the same straight line, and the four points not being in same plane, only one sphere, having the four points, in its surface, may be determined ; and the centre of this sphere is in the point of intersection between the three planes which bisect perpendicularly three straight lines joining one of the points to the three remaining points.

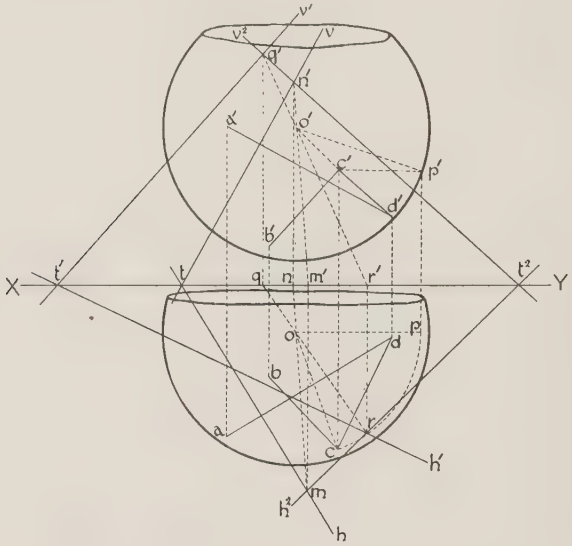


Fig. 188.*

Note.—Six straight lines may be drawn, each with its extremities in two of the points, and any three of these lines, not lying in the same plane, may be bisected perpendicularly by planes to give the centre of the sphere.

The above will be more clearly understood after considering the solution shown in Fig. 188. Join ad, cd, cb, a^1d^1, c^1d^1, and c^1b^1. Determine the plane vth to bisect AD perpendicularly, and a second plane $v^1t^1h^1$ to bisect CD perpendicularly. Then rq and r^1q^1 are the projections of the line of intersection of the two planes, and this line contains the centre of the sphere. For RQ is the locus of all points equidistant from the three points A, C, and D. Determine the third plane $v^2t^2h^2$ to bisect BC perpendicularly, and draw the projections of the line of

intersection nm, n^1m^1 between the planes vth and $v^2t^2h^2$. Then the point of intersection o between RQ and MN is the centre of the required sphere. For since MN also contains the required centre, being the locus of all points equidistant from the two points B and C and equidistant from the two points A and B, the point O is equidistant from the four points A, B, C, and D.

The radius of the sphere is determined by finding the distance of O from any one of the given points ; thus if op is made parallel to XY and equal to oc and p is projected to meet a horizontal from c^1 in p^1, o^1p^1 is equal to the required radius.

Problem 93. *Fig.* 189.—**Given the projections of a sphere and the positions of any two points on its surface, to determine the arc of the great circle of the sphere joining the points, and the poles of the great circle through the points.**

Let o and o^1 be the projections of the centre of the given sphere, pp^1 and qq^1 the projections of the given points. Join qp and q^1p^1. Then because QP is a chord of the great circle through P and Q, it is in the plane containing the required arc and the centre of the sphere, and therefore all radii of the required arc will intersect it.

Draw wx and w^1x^1 parallel to XY through o and o^1 respectively ; then wx is the plan of the elevation circle, and w^1x^1 the elevation of the plan circle, and the section of the arc between wx and XY in plan will be behind the solid in elevation, and the section of the arc between w^1x^1 and XY in elevation will be below the solid in plan. First, to find the points at which the arc passes behind the elevation circle and below the plan circle.

Project a to a^1 and draw o^1b^1 through a^1 ; b^1 being in the elevation circle. Project b^1 to meet ox in b. Then b, b^1 are the projections of the point of intersection between the elevation circle and the required arc. For A is the point in which the line PQ intersects the plane of the elevation circle ; and since PQ is in the plane containing the required arc and the centre of the sphere, and OB passes through A and is parallel to VP, B is in the required arc and in the elevation circle. From c^1, the point of intersection between o^1x^1 and p^1q^1, drop a perpendicular to meet pq in c and draw a straight line from o through c to meet the plan circle in d, and project d to meet o^1x^1 in d^1. Then OD is the horizontal radius common to the plan circle and the required arc. To find the projections of the point in which the required arc intersects the great circle whose plane is perpendicular to both planes of projection. Let yz and y^1z^1 be the projections of the great circle ; then j, j^1 are the projections of the point in which the chord PQ intersects the plane of the great circle, and J is in the common radius of this circle and the

required arc. Turn j about o to u and draw a projector from u to meet a horizontal from j^1 in u^1. Draw a straight line from o^1 through u^1 to meet the elevation circle in l^1 and project l^1 to meet ow in l. Then ol and o^1l^1 are the projections of the common radius rabatted into the plane of the elevation circle. From l^1 draw a horizontal to meet o^1z^1 in k^1, and turn l about o to k. Then K is a point in the required arc.

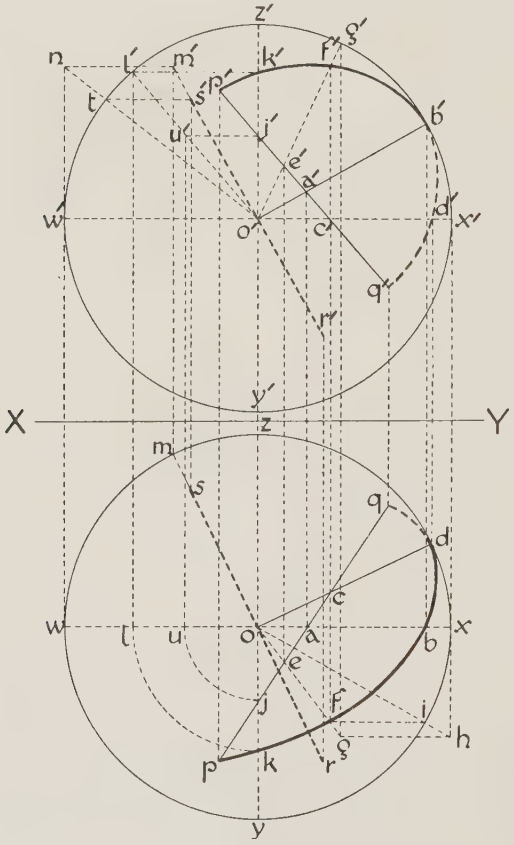


Fig. 189.*

Any number of points in the required arc may be found in the following way: Through any point c^1 in p^1q^1 draw a straight line from o^1 to meet the elevation circle in g^1. Project e^1 to e and drop a perpendicular from g^1 to meet oe produced in g. Draw gh parallel to XY and xh perpendicular to XY, and draw the straight line oh to cut the plan circle in i. Draw a parallel to XY from i to cut og in f and project f to f^1. Then f and f^1 are projections of a point in the required arc. For since

O and E are points in the plane of the required arc, the point in which the straight line OE produced intersects the spherical surface is in the arc ; and because G is in OE produced and oh is the true length of the line whose projections are og, o^1g^1, and oi is equal to the radius of the sphere, of and o^1f^1 are the projections of the radius through E, and F is in the arc.

The straight line through the centre of any plane section of a sphere, perpendicular to the plane of the section, intersects the surface of the sphere in two points called the *poles* of the section circle.

To determine the poles of the great circle through P and Q (same figure).

It has already been shown that o^1b^1 is in the plane of the great circle through P and Q and also in the vertical plane of the elevation circle ; then because the required poles are in the extremities of the diameter of the sphere perpendicular to the plane of the great circle through P and Q, the elevation of the diameter will be at right angles to the vertical trace o^1b^1. Also because od is the trace of the plane of the great circle PQ on the horizontal plane of the plan circle, the plan of the diameter containing the poles will be at right angles to od. Draw a perpendicular to od through o to meet the plan circle in m, and project m to m^1, making o^1m^1 perpendicular to o^1b^1. Produce ww^1 to meet a horizontal from m^1 in n, and draw no^1 to cut the elevation circle in t. Draw a horizontal from t to meet o^1m^1 in s^1, and project s^1 to s. Make or equal to os and o^1r^1 equal to o^1s^1. Then the required poles are S and R. The construction for finding s^1 is the same as that explained in connection with f, a point in the arc.

Trihedral Angles and Spherical Triangles.

The solid angle formed by three intersecting planes at their point of intersection is called a " trihedral angle " and comprises six elements such that any three being given, the remaining three may be determined. The six elements are the three plane angles formed by the lines of intersection of the three planes, and the three dihedral angles between the planes. If a circular arc be drawn to subtend the angle on each of the three planes, the three arcs being of equal radii and having the point of intersection of the three planes as their common centre, they will intersect in pairs on the lines of intersection between the planes and form a " spherical triangle ", and the six elements of the trihedral angle are said to be the elements of the triangle.

Consider Fig. 190, in which ORN and OR^1M are two right-angled triangles, each with an angular point in O, and the sides OR and OR^1

equal. Now suppose the triangle ORN to rotate about ON and OR^1M to rotate about OM ; then Rr is the plan of the locus of R and R^1r is the plan of the locus of R^1, therefore the line Or is the plan of the position in which OR and OR^1 coincide, and the two triangles together with the plane triangle ONM now form a trihedral angle about the point O. Let a circular arc, centre O, be drawn from R to R^1, cutting ON in P and OM in Q ; then because ORN and OR^1M are right angles, RN and R^1M are tangent to the arc. Let the arc QR^1 be turned about OM into the inclined plane of the triangle OrM ; then the plan of the arc will be an elliptic curve, tangent to Mr at r and to the circular arc at Q. In the same way the circular arc PR will turn about ON into the inclined plane OrN, and its plan will be the elliptic arc Pr, tangent to

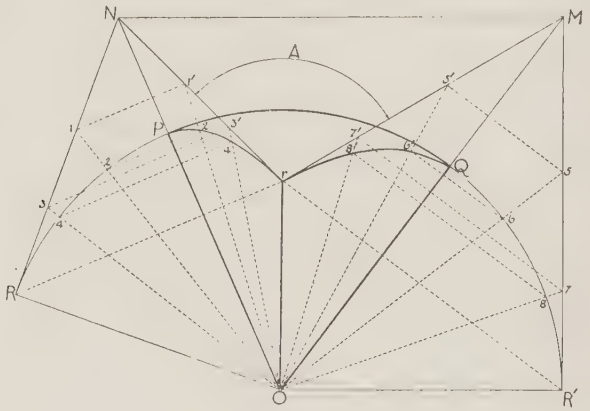


Fig. 190.

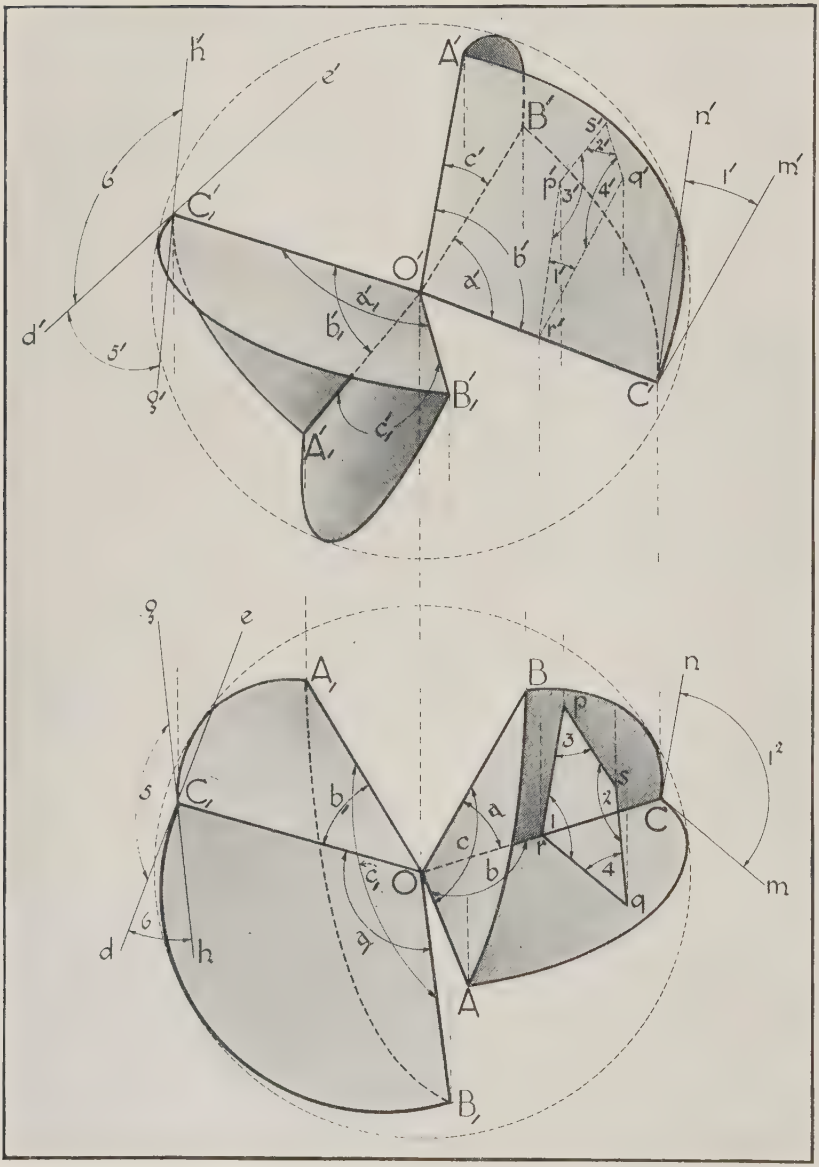
Nr at r and to the circular arc at P, and the three circular arcs now form a spherical triangle, the three arcs being parts of three great circles on the surface of a sphere whose centre is at O and whose radius is equal to each of the lines of intersection OP, OQ, and OR. The angles of the spherical triangle are the angles between the tangents at the extremities of the circular arcs, and these are equal to the dihedral angles between the sides forming the trihedral angle subtended by the triangle. For since Mr is a tangent to the arc Qr at r, and Nr is the tangent to the arc Pr at r, the real angle between Nr and Mr (not shown in the figure) is one angle of the triangle ; and because MR and NR are each perpendicular to the line of intersection OR, they contain the dihedral angle between the sides ORQ, ORP ; therefore A is the *plan* of the angle of the spherical triangle at R, and is the *plan* of the dihedral angle between the two sides containing R. The con-

ditions are the same for each of the two remaining angles, and the six elements are the three dihedral angles between the planes ORP, ORQ, OPQ, and the three plane angles between the lines of intersection OP, OQ, and OR. In subsequent references the former three will be called " angles " and denoted by the capital letters A, B, and C, the latter three will be called " sides " and denoted by the small letters a, b, and c; side a being opposite to angle A, side b opposite to angle B, and side c opposite to angle C.

Supplementary Trihedral Angles.

Fig. 191.—If from the point of intersection of three planes forming a trihedral angle a straight line be drawn perpendicular to each plane, the three planes each containing two of these lines will form a second trihedral angle such that each of its sides is the supplement of the opposite angle of the first trihedral angle and each of its angles is the supplement of the opposite side of the first trihedral angle. Thus, in Fig. 191 $OABC$ is the plan and $O^1A^1B^1C^1$ the elevation of a trihedral angle subtended by the spherical triangle ABC, $A^1B^1C^1$. Let OA_1, $O^1A^1_1$ be perpendicular to the side OBC, $O^1B^1C^1$; OB_1, $O^1B^1_1$ be the perpendicular to the side OAC, $O^1A^1C^1$; and OC_1, $O^1C^1_1$ perpendicular to the side OAB, $O^1A^1B^1$. Then the side a, a^1 is the supplement of the angle between the two sides intersecting in the line OA_1, $O^1A^1_1$; and similarly each of the six sides is the supplement of the angle between the two sides whose line of intersection is perpendicular to that side. For (considering the lower figure only), if the line rq in the side OAC is at right angles to OC and the line rp in the side OBC is also at right angles to OC, the angle qrp is the dihedral angle between the plane faces OAC, OBC. Let ps be perpendicular to the plane of OBC, and qs be perpendicular to the plane of OAC; then $qrps$ is a plane quadrilateral, since each of its sides is in the plane that cuts OC perpendicularly in r, and the four angles of the quadrilateral are together equal to four right angles. But the angles rps, rqs are right angles; therefore the angles prq, psq are supplementary; that is, angle $1 +$ angle 2 equals $180°$; and because OA_1 is parallel to ps, both being perpendicular to the plane of OBC, and OB_1 is parallel to qs, both being perpendicular to the plane of OAC, the angle c_1 between OA_1 and OB_1 is equal to the angle 2 between ps and qs, and because angles $1 + 2 = 180°$, angles $c_1 + 1 = 180°$; and since the angle 1 is the dihedral angle between the planes OAC, OBC, and c_1 is the side opposite that angle, and because the same relationship may be shown to exist between each side and opposite angle, the two trihedral angles are supplementary.

The quadrilateral is shown in elevation at $r^1p^1s^1q^1$.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b1
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c
2
C
c1
O
b
r
4
q
m
6
d
h
a1
A
B1

Fig. 191.*

Let AB, BC, and CA be arcs of great circles of a sphere, centre O; then ABC is a spherical triangle having for its elements the six elements of the trihedral angle it subtends. For since the angles of the triangle are the angles between the tangents at the extremities of the circular arcs which form the triangle, and these tangents are perpendiculars to the radii of the sphere, the angles between the tangents are equal to the angles between its sides; thus, Cm is at right angles to OC and is in the plane of OAC, and is therefore parallel to rq; and Cn and rp are parallel, being in the same plane and perpendicular to the same line; therefore the angles nCm, prq are equal.

Let A_1B_1, B_1C_1, and A_1C_1 be arcs of great circles of the sphere in whose surface are A, B, and C, then A_1 is the pole of the great circle through B and C, B_1 the pole of circle AC, C_1 the pole of circle AB, and A, B, and C are the poles of the circles through C_1B_1, A_1C_1, A_1B_1. The two spherical triangles are supplementary, and each triangle is said to be the polar triangle of the other.

The solution of a spherical triangle is obtained by solving the trihedral angle it subtends. There are six ways in which the six elements, taken three at a time, may be arranged for solution. These are as follows: Case I.—Given the three sides. Case II.—Given two sides and the included angle. Case III.—Given two sides and an opposite angle. Case IV.—Given the three angles. Case V.—Given one side and the two adjacent angles. Case VI.—Given one side, one adjacent angle, and one opposite angle.

The three cases IV., V. and VI. are usually solved by the aid of the polar triangle; thus, in case IV. each of the given angles is the supplement of one side of the polar triangle, therefore $180° -$ given angle $A =$ side a of the polar triangle, $180° -$ given angle $B =$ side b of the polar triangle, and $180° -$ given angle $C =$ side c of the polar triangle. The polar triangle may now be constructed with the sides a, b, and c, and its angles determined to give the sides of the required triangle. The author hopes that the direct solution given for case IV. will be found less tedious and more interesting than the above method.

Problem 94. *Fig.* 192.—**Given the three sides of a trihedral angle, to determine the angles.**

Let a, b, and c be the given sides. Arrange the given sides about any point O as shown, and with O as centre draw any circular arc to cut their outlines in points R, P, Q, R^1. Perpendicular to PO from R and perpendicular to QO from R^1 draw straight lines to meet in r, and join r to O. Draw a perpendicular to OR from R to meet OP produced in d, and produce OQ to meet a perpendicular to OR^1 from R^1 in e. With d as centre, dR as radius, and with e as centre, eR^1 as radius,

draw circular arcs to intersect on Or in n, and join n, d and n, e; then the angle dne (B) is the angle between the sides a and c. Draw a perpendicular to Rr at r, and with g, the point in which Rr intersects OP, as centre, radius gR, draw a circular arc to meet this perpendicular in R^3, and join R^3g; then the angle R^3gr (C) is the angle between the sides a and b. Make R^2r equal to R^3r and perpendicular to R^1r. Join R^2 to h. The point of intersection between Oe and R^2hr (A) is the angle between the sides c and b.

Case II., in which two sides and the included angle are given, requires in its ordinary form nothing beyond the drawing of a given triangle to make a given angle with the plane of the paper. When, however, an

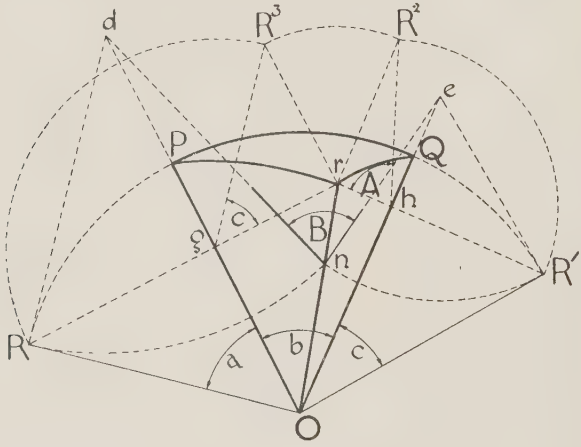


Fig. 192.*

additional condition is introduced, as in the following problem, the construction is more interesting.

Problem 95. *Figs.* 193 *and* 194.—**Given the two *inclined* sides of a trihedral angle and the angle between them, to determine the third side and draw the plan of the angle.**

Let a and c be the given sides and B the given angle. Arrange a and c about any point O, as shown in Fig. 193. Draw any line pq perpendicular to Or, the line of intersection between the given sides.

Draw rq^2, equal to rq, to make the given angle B with rp. Draw q^2p, and perpendicular to it draw sr. With r as centre, rs as radius, draw a circular arc to cut pq in n. With O as centre, On as radius, cut Or produced in m. Draw a perpendicular to or through m to meet circular arcs through p and q, centre O, in p^1 and q^1. Draw p^1O and q^1O. Then q^1Op^1 (b) is the required side. If the circular arc PQ through

n and m is the side of the spherical triangle subtending the side b, the other two sides will intersect in r.

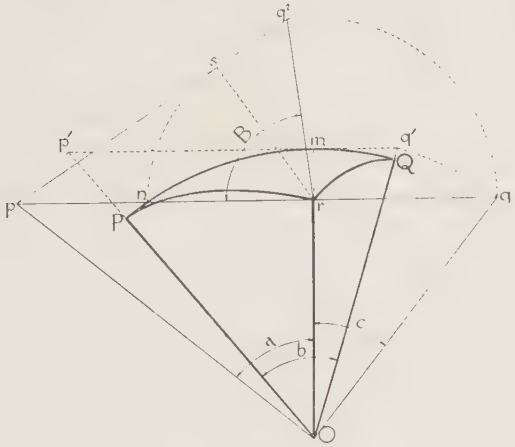


Fig. 193.

Explanation of the Foregoing Construction.

Fig. 194.—As before, let a and b be the given sides and B the given angle, and let om^1 be the line of intersection.

Then, because pm and qm are lines in the given sides perpendicular to the line of intersection, the angle between these lines, when the sides are raised into the required positions, will equal the given angle B. Now suppose the line of intersection Om to turn upwards about the point O, moving always in the vertical plane on Om; then, if the lines Op, Oq move always in the plane of the paper so as to keep the distances pm, qm constant, m being in the moving line of intersection, the loci of p and q will be circular arcs pp^1, qq^1 centred in O. Now consider the triangle pmq^2, in which mq^2 equals mq and the angle pmq^2 equals the given angle B. When the given sides are in the required position and the lines pm, qm contain the given angle B, the two lines, together with the straight line joining p and q, will form a triangle equal in every respect to the triangle pmq^2, having the side corresponding to pq^2 in plane of the paper and perpendicular to Om. For, because pm and qm are then perpendicular to Om, the plane containing them will be perpendicular to Om, and therefore its horizontal trace will be perpendicular to the plan of Om.

Let the triangle pmq^2 be turned about the point m so as to bring the side pq^2 perpendicular to Om, as shown at p^2q^3, then because ms is at right angles to pq^2, s moves to s^1 in Om produced.

Consider, now, only the three plane triangles, pmO, qmO, and q^3mp^2, in which $qm = q^3m$, $pm = p^2m$, qmO and pmO are right angles, p^2mq^3 equals the given angle, p^2q^3 is perpendicular to Om, and ms^1 is in Om produced. Let the point m, containing an angular point of each of the three triangles, be raised about O so as to have the line of intersection Om and the line ms^1 always in the vertical plane on Om, as already explained, and the lines Op, Oq, and p^2q^2 always in the plane of the paper. Then m will describe a circular arc, centre O, radius Om, in this vertical plane; the arc being shown rabatted at mm^2, p will move in the arc pq^1, q will move in the arc qq^1, p^2 will move in the straight line p^2p^1, and q will move in the straight line q^3q^1; and when the points p and p^2 coincide in p^1, and the points q^3 and q coincide in q^1, m will have moved through the arc mm^1 (rabatted), where mm^2 equals ms^1, and the plan of m will be in m^3, the foot of the perpendicular to Om from m^2. Because the line m^3s^2 is in the plane of the triangle $p^1q^1m^3$, and p^1m^3 and q^1m^3 are perpendiculars to the line of intersection, the straight line through Om^2 is the rabatment of the line of intersection and is perpendicular to m^2s^2, and if m^2 is turned about s^2 to m^1, $p^1m^1q^1$ must equal the given angle B. Then q^1Om^3 is the required plan of the side c, p^1Om^3 the required plan of the side a, and b is the third side of the trihedral angle. With centre O, radius Os^2, draw a circular arc to cut pq in n. Then, because n is in Om^2 produced, s^2 in Om produced, and m^2n equals s^2m, the line mn equals the line s^2m^2; and since s^2m^2 equals ms and p^1s^2 is a tangent to the arc ns^2 at s^2, the construction may be reduced to that shown in Fig. 193, as explained in problem 95. Compare the two figures.

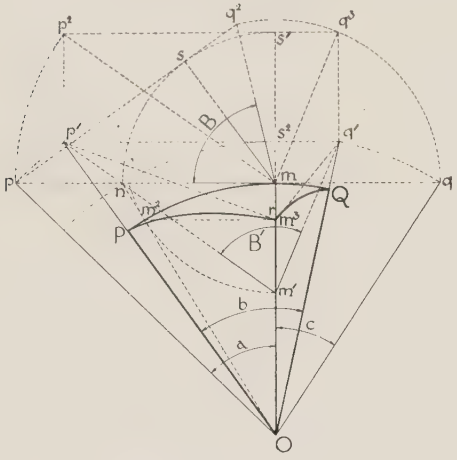


Fig. 194.

Case III. *Fig.* 195.—Given two sides and an opposite angle. Let a and b be the given sides, and B the given angle.

Arrange a and b about any point O as shown. Draw a perpendicular to On from any point m in the line of intersection between the given sides to meet On in s. Draw sm^1 to make the given angle B with sm,

PLATE IV.

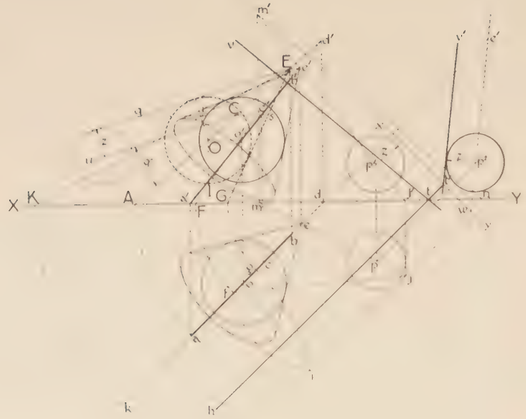


FIG. 180.*

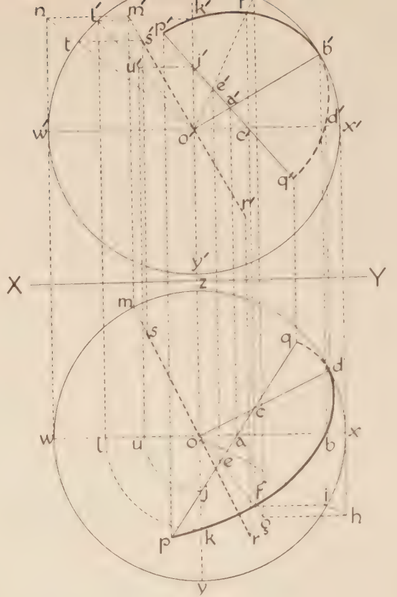


FIG. 189.*

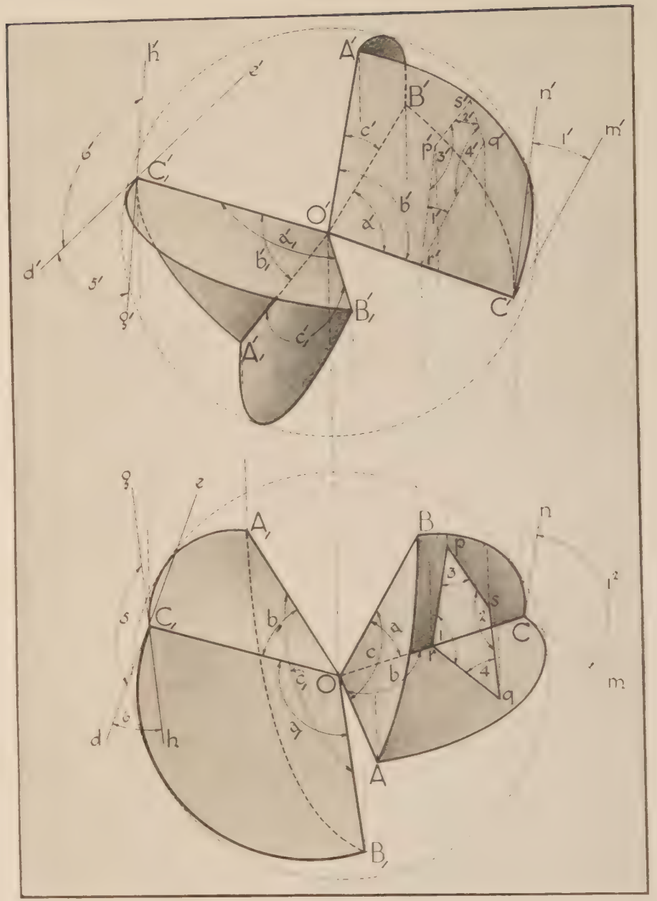


FIG. 191.*

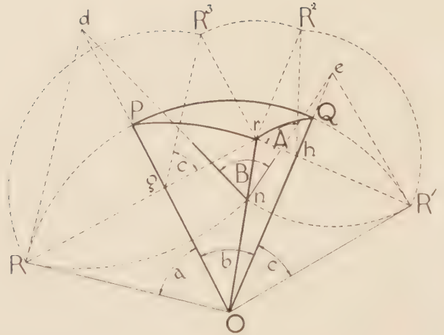


FIG. 192.*

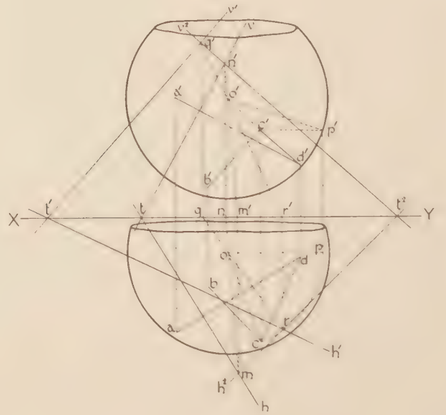


FIG. 188.*

Face page 176.

making mm^1 perpendicular to sm. Produce Om to m^2, making mm^2 equal mm^1. Draw nR perpendicular to Om through m, and join m^2, n. With m as centre, mR as radius, draw a circular arc to cut nm^2 in r^1 and r^2. From r^1 or r^2 draw a line to meet nR perpendicularly in r. Join OR. Then nOr is the plan of the third side, c, of the required trihedral angle. The plan of the subtending spherical triangle may now be drawn as at PQr, and the side c may be rabatted by turning r^1 about n to meet the arc, centre O, radius OR, in R^1. There is a second solution obtainable by drawing the perpendicular to nR from r^2; therefore Case III. is the ambiguous case. When the arc Rr^1r^2 is tangent to nm^2, only one solution is possible.

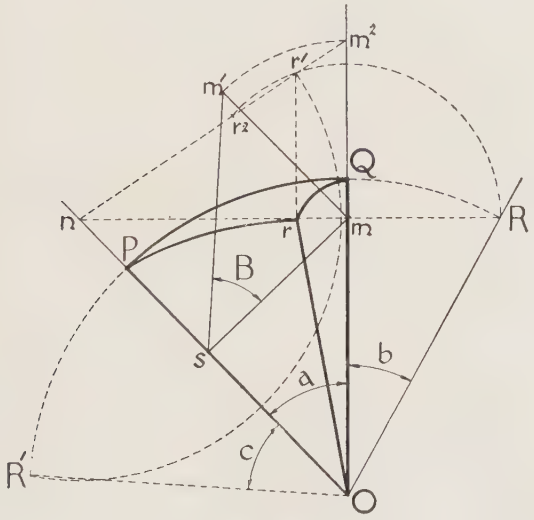


Fig. 195.

Note.—The triangle nm^2m is a rabatment of the section of the planes of the sides a and c containing the angle B, and a vertical plane on Om, by a second vertical plane perpendicular to Om; and since the locus of R is the arc Rr^1r^2 and is in the section plane when the side b is turned about Om, either r^1 or r^2 is a rabatment of the point of intersection of the three planes.

Case IV. Fig. 196.—Given the three angles of a trihedral angle, to determine the sides.

Let A, B, and C be the given angles.

Draw fe to make one of the given angles (A) with any straight line XY, and draw fO perpendicular to XY. Then, if one of the

required sides (*b*) is in the plane of the paper, and a second side in the plane *efO*, their line of intersection will be in *fO* and the two sides will contain the given angle *A*. From any point *d* in *fe* draw a straight line to make a second given angle (*B*) with *fe* and meet *XY* in *v*. Draw *hv* perpendicular to *fe*.

Regard *hv* as the axis and *dv* as an outline generator of a right circular cone having its base in the plane *efO* and its vertex in the plane of the paper. Then any plane to touch this cone will make the given angle *B* with the plane *efO*, and *v* will be a point in its

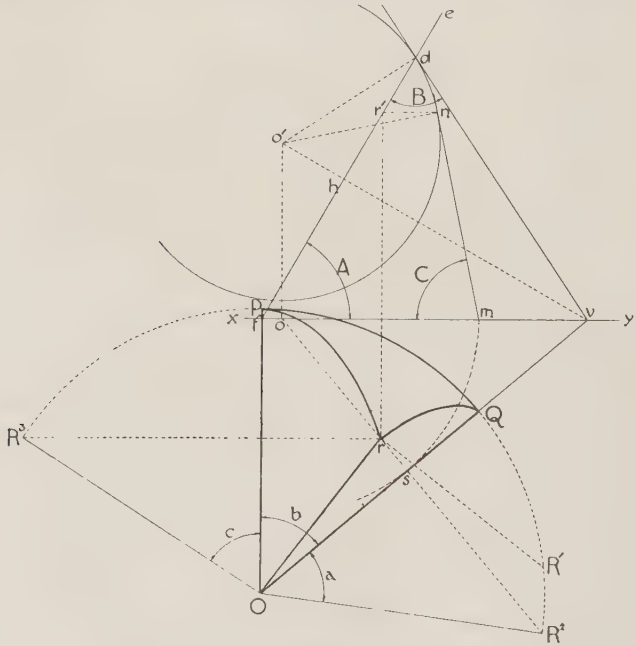


Fig. 196.

horizontal trace. Draw a perpendicular to *vd* from *d* to meet *vh* produced in o^1, and with o^1 as centre draw a circular arc through *d* to represent a sphere inscribed in the cone *vdh*. Draw *mn* to make the third given angle *C* with *xy* and touch the outline of the sphere in *n*. Draw nr^1 parallel to *xy*. Then, because the generator *vd* touches the sphere at *d* and dr^1 is at right angles to the axis *vh*, r^1 is the elevation of a point in the circle of contact between the cone *vdh* and the sphere ; and because *mn* touches the sphere at *n* and nr^1 is parallel to *xy*, r^1 is the elevation of a point in the circle of contact between the sphere and a cone standing with its base in the plane of the paper, and its

base angle equal to the given angle C. Because the two cones vd and mn envelope a common sphere, a plane may be found to touch them, and since R is the point of intersection between the contact circles, the tangent plane will contain R, and because R is also in the plane Ofe, it is in the line of intersection of the two planes. Draw a perpendicular from o^1 to meet xy in o, and with o as centre, om as radius, draw a circular arc to represent part of the base circle of the cone mn, and draw a tangent to this arc from v, touching the arc in s and intersecting fO in O. Join os, and draw a projector from r^1 to meet os in r; join Or. Then fOv (b) is one side of the trihedral angle, and vOr, fOr are the plans of the remaining sides. Draw rR^1 perpendicular to Or and equal to the height of r^1 above xy. With O as centre, OR^1 as radius, draw a circular arc to cut os produced in R^2, and a perpendicular to Of from r in R^3. Join O, R^2 and O, R^3 to obtain the sides a and c. The plan of the spherical triangle to subtend the angle may now be determined as shown.

Note.—The vertex v of the cone vd and the base of the cone mn are both in the plane of the side b. Also, R is in the plane of the side c in the base circle of the cone vd and in the surface of the cone mn. Then, because the plane of the side a contains v and R and touches the base of the cone mn, it touches both the cones and therefore makes the given angle C with the side b and the given angle B with the side c.

Problem 96. *Fig.* 197.—**To determine the approximate development of the surface of a sphere.**

The surface of a sphere cannot be developed into a plane surface, but there are two methods of obtaining an approximate development shown in the figure. On the left of the centre line, oo^1, the surface is divided by a series of circles lying in parallel planes into areas known as zones, and by regarding each zone as a conical surface the development is obtained as follows: Consider the surface shown at $abcd$ in plan and $a^1b^1c^1d^1$ in elevation.

Draw a straight line through d^1c^1 to meet oo^1 produced in v. With v as centre, vd^1 and vc^1 as radii, draw the circular arcs d^1b^2 and c^1a^2, making d^1b^2 equal to db or c^1a^2 equal to ca. Draw a straight line from v through a^2 and (or) b^2; then $d^1c^1b^2a^2$ is the development of the zone or area $ABCD$. Each zone may be developed in the same way as shown in the figure.

Note.—In large domes where the surface is covered by a great many narrow strips or battens the area nearer the base of the dome does not require development, since the battens may be bent in both directions to the required curvature. The uppermost area is usually

formed by a disc, known as a boss, turned on a lathe to the required shape.

On the right of the figure the area is divided by great circles of the sphere lying in vertical planes into areas known as lunes. Lunes are developed as follows: Let *onm* be the plan of a lune symmetrical about *ot*. Divide the elevation curve t^1u into any number of equal parts, and project the points of division f^1e^1, g^1h^1, j^1k^1 to meet the plan lines *om* and *on* in points *f*, *e*, *h*, *g*, *k*, and *j*. Produce *ot* to *O*, making *tO* equal to the stretch-out of the quadrant t^1u. Divide *tO* into four equal parts to correspond with the division of t^1u, and draw *JK*, *GH*, and *EF* perpendicular to *tO* amd equal to *jk*, *gh*, and *ef* respectively.

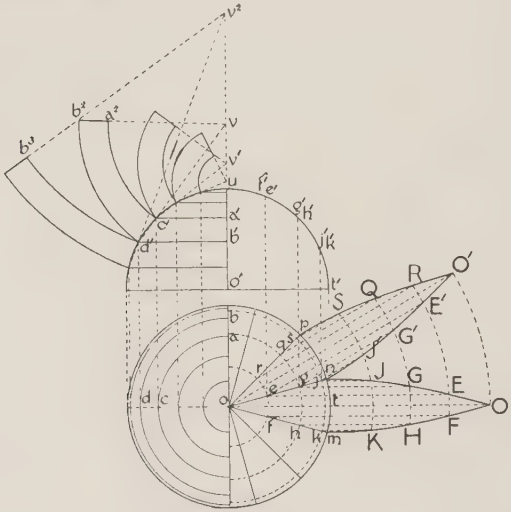


Fig. 197.*

Curves through *n*, *J*, *G*, *E*, *O* and *m*, *K*, *H*, *F*, *O* enclose the development of the lune *onm*. By drawing circular arcs about *o*, other lunes may be developed as shown.

Problem 97. *Fig.* 198.—**To draw the half plan, half inverted plan, half elevation, and half sectional elevation of a hemispherical stone dome, given the internal radius (*z*), the thickness of the base (*x*), and the thickness at the crown (*y*).**

Draw the quarter circle p^1r^1 of the given radius *z*, and centre m^1. Produce m^1p^1 to q^1, making p^1q^1 equal to the given base thickness *x*, and produce m^1r^1 to e^1, making r^1e^1 equal to the given crown thickness *y*. Make m^1s^1 equal to m^1q^1 and draw the circular arc $q^1e^1s^1$. Draw the plan circles *sqn* and *mp*. Divide the outline curves in plan and elevation

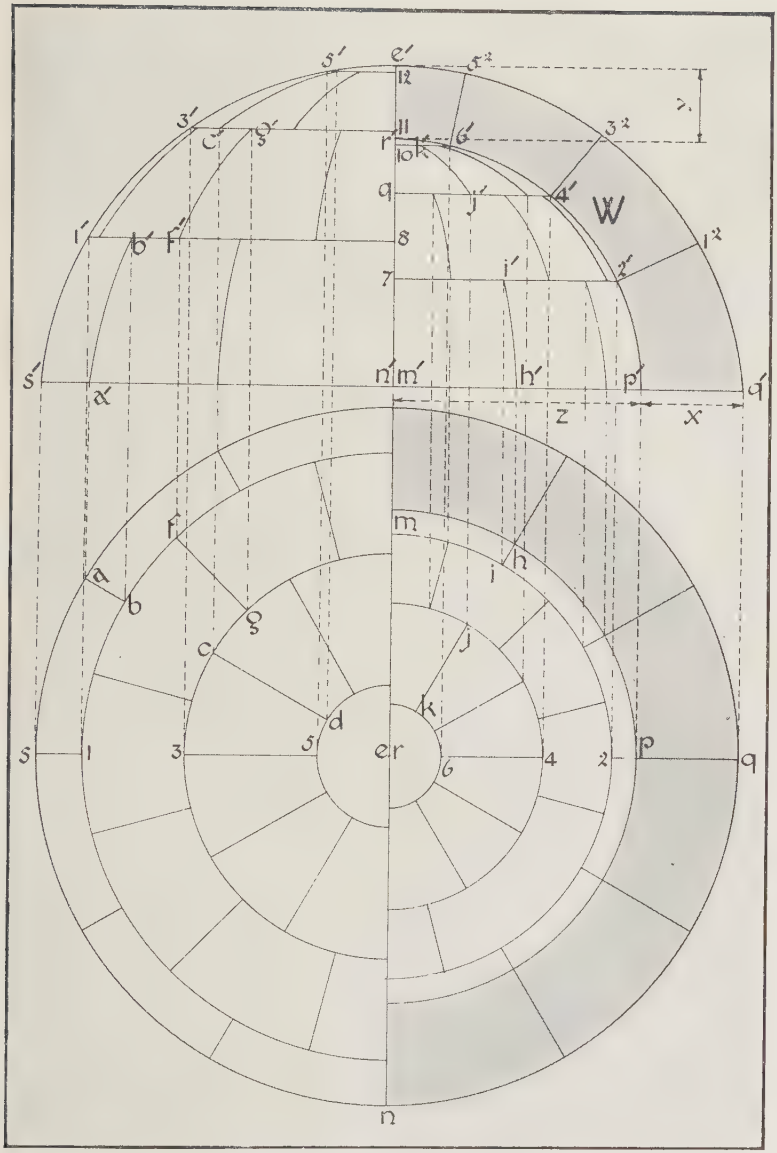


Fig. 198.

into the required number of courses, and from the points 1^2, 3^2, 5^2 draw radials from m^1 cutting the arc p^1r^1 in points 2^1, 4^1, and 6^1. Draw 2^17, 4^19, 6^110, 1^18, 3^111, 5^112 parallel to s^1q^1. Project 2^1, 4^1, 6^1, 1^1, 3^1, 5^1 to meet sq in 2, 4, 6, 1, 3, 5 and draw the plan circles through the latter six points. The plan may now be completed by drawing the radial joints *ab*, *cd*, *jk*, *hi*, etc. etc. The inner semicircles *ed*, *rk* in plan are half plan and half inverted plan of the boss, and the course immediately below it would, ordinarily, have alternate vertical joints omitted. The elevations of the vertical joints are elliptic arcs and may be trammelled or projected; in either case it is advisable to draw the full quarter ellipse and erase the portions not required.

The natures of the joint surfaces are fully explained below in connection with the voussoir marked " W ".

The Geometry of the Voussoirs in Hemispherical Domes.

Figs. 199 *and* 200.—In Fig. 199 an enlarged plan and sectional elevation of the voussoir W (Fig. 198) are shown at aa^2gg^2 and $a^1b^1c^1d^1e^1f^1g^1h^1$.

It will be seen that the voussoir is a solid bounded by two plane surfaces, two spherical surfaces, and two conical surfaces. The two plane surfaces are in the vertical joint planes and each of these surfaces is bounded by two circular arcs of radii equal to the inner and outer radii of the dome, and two straight lines which radiate to the centre of the dome. The section $b^1d^1f^1h^1$ is the true shape of each of these plane surfaces. The two spherical surfaces lie one in the inner surface and one in the outer surface of the dome, and each is bounded by four circular arcs, two of the arcs being equal and each a part of a great circle of the sphere containing the surface; the other two are unequal and are arcs of small circles of the sphere containing the surface. The two conical surfaces are each bounded by two circular arcs and two straight lines. Thus, the four straight arrises of the voussoir are normals to the inner face of the dome and are the lines of intersection between the plane surfaces and the conical surfaces. The four arcs of great circles—two in the inner and two in the outer face of the dome—are the lines of intersection between the two spherical surfaces and the two plane surfaces. The four arcs of small circles—two in each face of the dome—are the lines of intersection between the two spherical and the two conical surfaces.

Let the straight line through a^1e^1 be the edge view of a plane containing the four solid angular points of the voussoir which lie in the inner face of the dome, and let i^1p be an edge view of a parallel plane tangent to the outer face of the dome. Then if the two plane surfaces

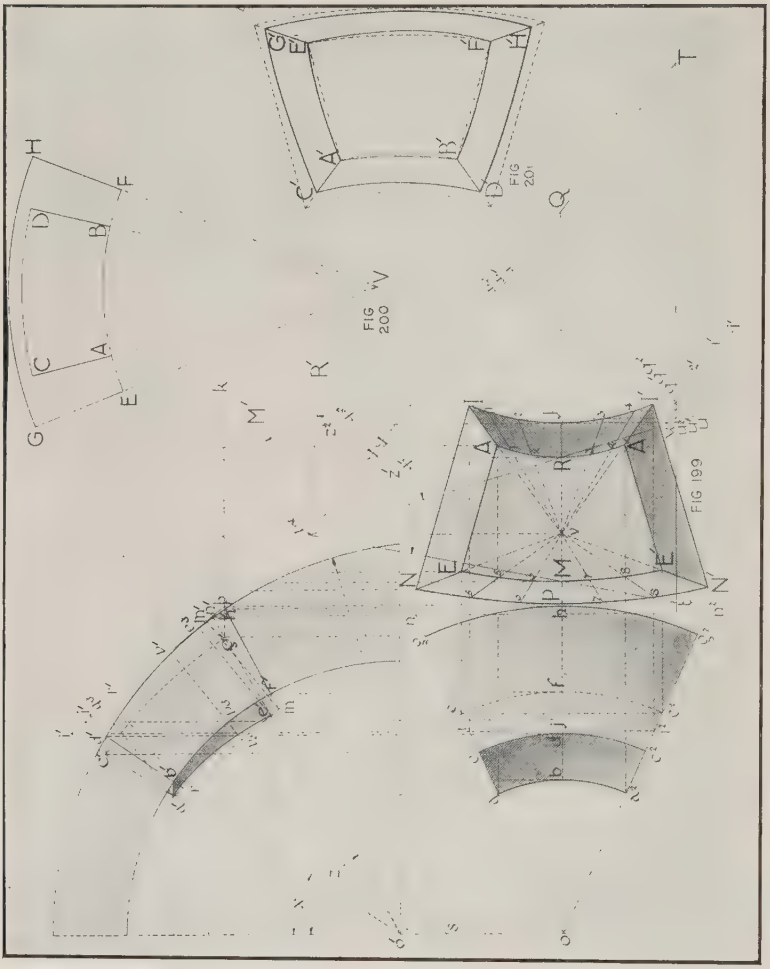
and the two conical surfaces be produced to cut these planes, the lines of intersection on the planes will be the outlines of two moulds, such that if the moulds are placed in their proper relative positions on two parallel faces of a block of stone, whose thickness is equal to the distance between the planes, the stone may be worked to the moulds so as to obtain the two plane surfaces and the two conical surfaces. If, now, the conical surfaces be developed to obtain flexible moulds, and these moulds, together with two flat moulds shaped as $b^1d^1f^1h^1$, are applied, the bounding lines of the two spherical surfaces may be scribed on the stone.

Because the lines of intersection between the plane and conical surfaces are straight, they are parts of generators of the inverted cones whose surfaces contain the conical joints ; and because these lines all radiate to the centre of the dome, the vertex of each cone is in the centre of the dome ; and by projecting the common vertex of the cones perpendicularly on to the face moulds AE and NI, the point in the inner spherical surface most distant from the face of the stone is determined. Each of the face moulds for this particular voussoir is bounded by two straight lines, one elliptical arc, and one hyperbolic arc. The natures of these arcs are, of course, dependent upon the angles at which the face mould planes cut the cones, and may be elliptic, hyperbolic, or, occasionally, parabolic.

Method of obtaining the Moulds.

Fig. 199.—Draw the straight line through a^1e^1 and draw i^1p parallel to a^1e^1 and tangent to the outer curve. Produce, where necessary, the straight lines a^1c^1, b^1d^1, e^1g^1, and f^1h^1 to meet a^1e^1 in a^1, e^1, r, and m, and to meet i^1p in i^1, j^1, n^1, and p. Project a^1, r, e^1, and m to meet i^1p perpendicularly in $a^3r^1e^3$ and m^1. Project i^1 and n^1 to meet og in i and n, and og^2 in i^2 and n^2. Draw the horizontal line pk, and with p as centre turn into it all the points in i^1p and project them to meet horizontals from a, i, e, and n in A, I, E, and N, and from a^2, i^2, e^2, n^2 in A^1, I^1, E^1, and N^1, and from o in P, M, R, and J. Then N, N^1, I, I^1, P, and J are points in the outside mould, NI and N^1I^1 being straight lines, NPN^1 a hyperbolic arc, and IJI^1 an elliptic arc. The inner mould E, E^1, A, A^1, M, R is a similar figure, EA and E^1A^1 being straight, EME^1 hyperbolic, and ARA^1 elliptic. P, M, R, and J are the vertices of the conics, and the curves may be determined in full-sized work by drawing any number of radials from o^1 between a^1r and e^1m to cut a^1m and i^1p, finding the plans of the points of intersection, and developing in the same way as the points N, I, etc.

To obtain the directions of straight lines across the conical joint surfaces. Draw o^1v^1 perpendicular to a^1e^1. Turn v^1 about p into pk and project to meet the horizontal from o in v; then any straight line from v, as 11, 22 . . . 88, is part of a straight generator across the joint



FIGS. 199, 200, and 201.*

surfaces, and by marking the numbered points on the moulds the directions of straight drafts may be set out on the stone. Draw vv^4 perpendicular to MR and equal to o^1v^2, and join v^4M and v^4R. Then the angle Mv^4R equals the angle mo^1r. Draw v^4Q to make the angle z with v^4M or to make the angle x with v^4R, that is, make the angles

$z^1 = z$ and $x^1 = x$, and repeat these angles on the other side of v^4Q, as shown. Then v^4Q is the common axis of the two cones whose surfaces contain the conical joint surfaces; v^4M, v^4M^1 the outline generators of one of the cones; v^4R, v^4R^1 the outline generators of the second cone, all rabatted into HP about the line MR. Draw Pt perpendicular to MR and equal to v^1v^2, the thickness of the stone block, and draw tT parallel to MR to cut v^4R in w, and produce v^4M to t. Then Mt is the real length of PM, and the angles between the straight draft PM and the faces of the stone are equal to the angles between MR and v^4t. Similarly, Rw is the real length of RJ and the angle MRJ equals the angle MRw.

Points in any of the four curves may now be found as follows: Draw any ordinates between I and J to meet the section plane tT in points q, q^1, q^2. Draw yu, y^1u^1, y^2u^2 through q, q^1, q^2 perpendicular to the axis v^4Q. With y^2 as centre, y^2u^2 as radius, draw a circular arc to cut a perpendicular to y^2u^2 from q^2 in 2^1; then q^22^1 is half the length of the ordinate 23, and is the distance to set out on both sides of JQ along the ordinate from q^2. Any number of points may be found in each arc by the same process.

To determine the flexible moulds for the conical joint surfaces (Fig. 200). With any point V as centre, radii o^1b^1, o^1d^1, and o^1h^1 (Fig. 199), draw the circular arcs EF, CD, and GH (Fig. 200). Make the arc EF equal to the arc ee^2 (Fig. 199), the arc AB equal to the arc aa^2 (Fig. 199), and draw straight lines from V to obtain GH and CD; then $ABCD$ is the mould for the upper conical joint surface and $EFGH$ the mould for the lower surface. Revert to Fig. 199. If the four joint moulds be scribed on the worked joint surfaces, the moulds for the vertical joint planes being shaped to the form of $b^1d^1f^1h^1$, as already explained, and each mould having the extremities of its concave edges in the face AA^1EE^1, and if a sinking is made in the latter face at the position v equal to the distance v^2v^3, a convex template shaped to the great circle of the inner surface may be applied always through v to assist in working the concave spherical surface. The point of contact between the outer spherical surface and the plane face of the stone may be found by drawing a perpendicular to ip from s, and a concave mould may be applied through this point when working the outer spherical surface. The templates shaped to the great circles will conform to the finished spherical surfaces in any direction if held in a normal position, but this cannot be done readily until the surfaces are worked. When the templates are applied through the point v and the contact point, the flat faces of the templates are perpendicular to the plane faces of the stone, and curved drafts

may be worked through the points by applying the templates in this position.

In Fig. 201 the positions of the moulds relative to each other are shown. The dotted lines indicate the moulds AEA^1E^1 and NIN^1I^1 (Fig. 199). The flexible moulds for the conical joint surfaces are lettered to correspond with Fig. 200. The moulds on the plane joint surfaces are $C^1G^1A^1E^1$ and $B^1F^1D^1H^1$. The firm lines representing the positions

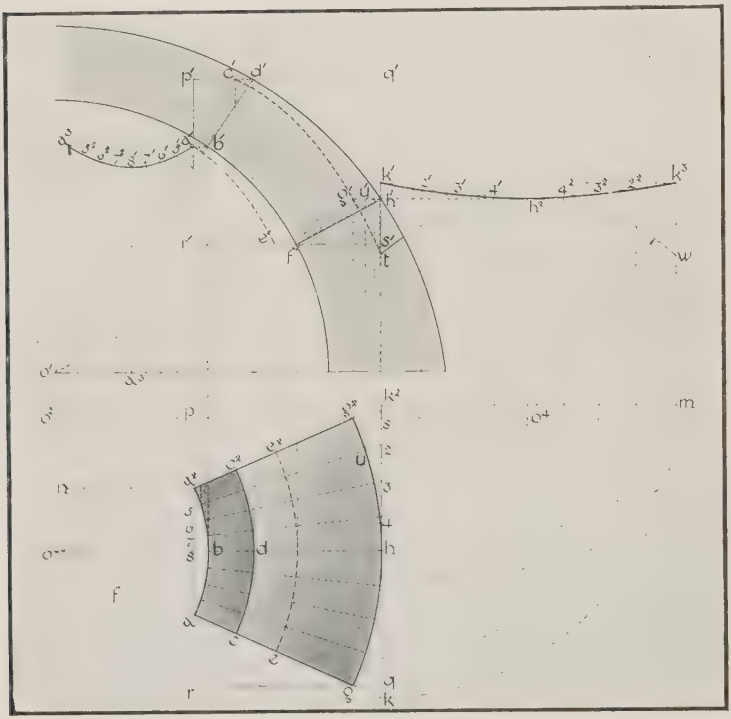


Fig. 202.

of the four joint moulds may also be regarded as a plan of the finished voussoir lying with its convex surface tangent to the plane of the paper.

In Fig. 202 a more simple but less economical method of working the voussoir is shown. The larger block of material required is shown in position about the voussoir at $pqrs$ in plan, and $p^1q^1r^1s^1$ in elevation. It will be seen that the conical joint surfaces produced meet the vertical faces PR and QS in hyperbolic curves, and meet the horizontal faces PQ and RS in circular arcs. The circular arcs are the plan curves cc^2 and ee^2, and are shown in their true positions on the horizontal

surfaces. The hyperbolic arcs are determined as follows: Produce ag and a^2g^2 to meet sq produced in k and k^2. From o, the plan of the centre of the inner spherical surface, draw radials across the plan to meet pr in points 5, 6, 7, 8, etc., and sq in points 2, 3, 4, h, etc. Project the points in which these radials meet the plan curves aba^2 and ghg^2 into a^1b^1 and g^1h^1 and draw the elevations of the radials to meet p^1r^1 and q^1s^1. From the points of intersection between the elevations of the radial lines and the lines p^1r^1 and q^1s^1 draw horizontal projectors of indefinite length. Draw k^2m and na^2 perpendicular to sq and pr and turn the points 2, 3, 4 . . . k about k^2 into km, and turn the points 5, 6, 7 . . . a about a^2 into na^2. Projectors from the points in k^2m and na^2 meet the corresponding horizontal projectors in points on the required curves. The scale of the drawing is too small to allow every point being lettered, but the procedure for each point in the curves is the same as shown at $u2$ and u^12^1 where o^1u^1 is the elevation and ou the plan of one generator.

The plane joints may be worked from moulds shaped to the plan outline aa^2gg^2, and the spherical surfaces from the flexible moulds and vertical joint moulds exactly as in the previous example.

The circular arcs a^3a^1 and tw, centres o^3 and o^4, are the lines of intersection between the spherical surfaces and the vertical surfaces of the block, and by scribing these arcs on the material, together with arcs cc^2 and ee^2, the spherical surfaces may be worked independently of the conical joint moulds.

Spherical Pendentives.

Fig. 203.—The isometric sketch, Fig. 203, shows the geometrical nature of a hemispherical dome with a circular void and four pendentives.

In a practical form the circular void would be covered by a circular sky-light, lantern light, or turret, and the pendentives would spring from vertical wall surfaces. The semicircles BFA and AED are vertical, plane surfaces; and the spherical surface lying between these semicircles and the dotted arc FE is called a pendentive, FE being part of the plane section of the sphere by a horizontal plane tangent to the semicircles at F and E.

In this example there are four pendentives, one from each of the springing points A, B, C, and D, but there may be any number according to the number of vertical planes arranged to cut the hemisphere.

In wooden-framed domes of this type there are no geometrical difficulties, since the ribs are radiating and therefore of equal curvature, only the lengths and bevels being variable.

In Fig. 204 the part plan and sectional elevation of a wooden-framed

pendentive dome are given, showing how the elevations, lengths, and bevels of the ribs may be obtained. The diameter of the dome is

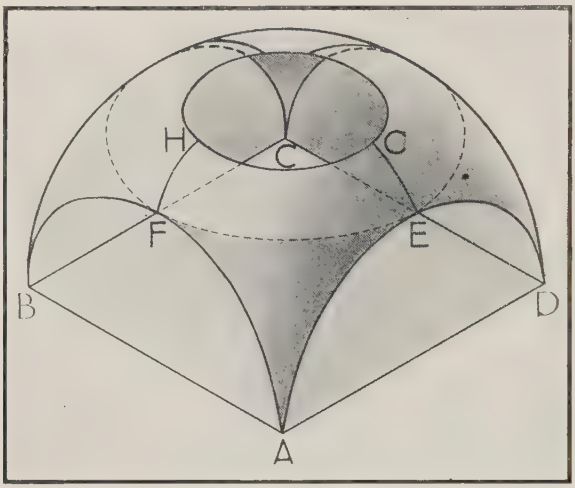


Fig. 203.

equal to the diagonal of the square chamber covered by the dome. On each of the four vertical wall surfaces a semicircular plate as shown

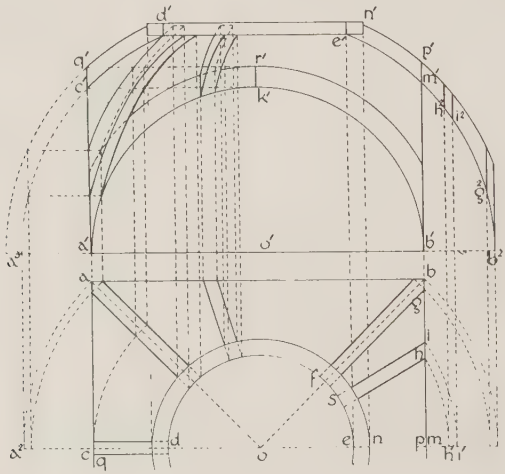


Fig. 204.

at $a^1k^1b^1$ is fixed, and the lower ends of the ribs are bevelled to fit it. The upper ends of the ribs are cut to intersect the circular curb *def*, as

shown at c^1 and d^1, and because the ribs radiate towards the centre of the curb, the upper ends are all shaped alike.

When the plan is given, the elevation is obtained as follows: With o as centre, oa as radius, draw an arc to cut the centre line dc produced in a^2. Draw a^1b^1 horizontal and produce to meet a projector from a^2 in a^3. Project o to o^1, and with o^1 as centre draw a semicircle to intersect perpendiculars from a^1 and b^1 in c^1 and m^1. Project the curb to cut this semicircle as at e^1 and n^1. Set up the thickness of the curb and draw the upper edges of the short ribs, as n^1p^1, to represent any given width of the ribs. Draw the semicircle $a^1k^1b^1$ and draw a parallel arc through r^1, making k^1r^1 equal to m^1p^1 or c^1q^1. Any number of points in the elevations of the ribs may be obtained by drawing circles about o across the plan, finding the heights of these circles by projecting their points of intersection with the centre line cd, produced if necessary on to the elevation circles through c^1 and q^1, and making all the points on the same plan circle equal in height. The figure should explain this clearly.

The lengths and bevels of the ribs are found as shown on the right of the figures. Each rib is turned about o into the centre line and projected to cut the elevation curves produced; thus, hi is turned to h^1i^1 and projected to h^2i^2, then e^1i^2 is the overall length of the rib sh, and the difference in the lengths of its vertical faces is equal to the distance between the lines h^2 and i^2.

In Fig. 205 the half plan, half inverted plan, half elevation, and half sectional elevation of a stone pendentive dome are given.

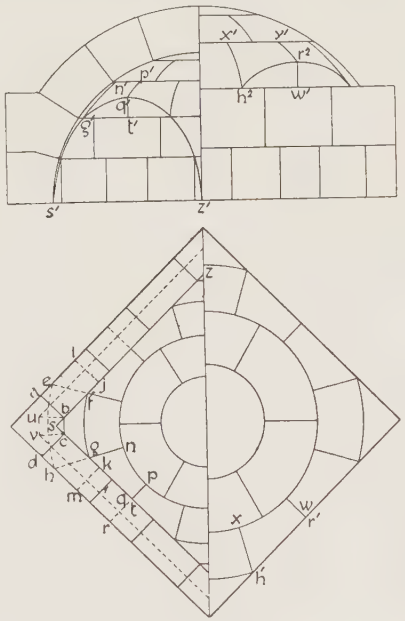


Fig. 205.

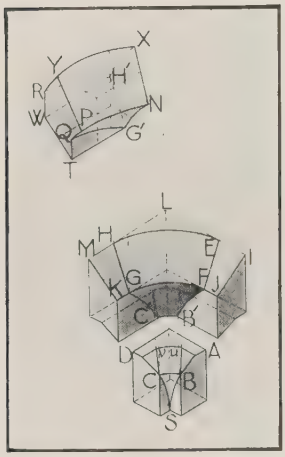
The semi-ellipse $s^1q^1z^1$ is the elevation of one of the plane semicircular faces between the pendentives.

The author has a somewhat limited knowledge of practical masonry, and the construction of the dome is arranged to provide some useful

geometrical problems. The dome may perhaps have some structional imperfections, but any change in the voussoir arrangement would involve the same geometrical considerations.

The voussoirs forming the pendentives are shown in the inverted plan at *abcd*, *ijkmfgbc*, and *pnhrgq*; there are two of the latter about each pendentive.

In Fig. 206 the voussoir containing the springing point *S* of the pendentive is shown enlarged in isometric projection, and the capital letters correspond with the small letters in Fig. 205. The lower bed of this voussoir is a plane horizontal surface. The upper bed comprises one plane horizontal surface, two inclined-plane triangular surfaces, and one conical surface. The conical surface is bounded by two circular arcs *CB* and *vu*, and two straight lines *uB* and *vC*. The inner angle of the voussoir is worked to form a part of each of the two adjacent plane semicircular surfaces, and part of the spherical surface; the latter is bounded by the three circular arcs *CB*, *CS*, and *BS*.



Figs. 206, 207, and 208.

The voussoir immediately above the one just described is shown in Fig. 207, and the positions of the two voussoirs relative to each other are indicated by dotted lines.

The lower bed of 207 is worked to conform to the upper bed of 206, the voussoir being made large enough to project beyond the lower voussoir, as shown by the dotted lines in Fig. 205. The inner face, C^1B^1GF, of 207 is a continuation of the spherical pendentive surface, the arcs C^1G and B^1F being each a part of one of the semicircular lines of intersection between the pendentives and the vertical plane wall surfaces; therefore the faces KC^1 and JB^1 are parts of the latter surfaces.

The upper bed consists of a conical surface *EFGH*, *HE* and *GF* being circular arcs, *HG* and *EF* straight lines, one plane horizontal surface *LEH*, and two inclined plane surfaces *MHGK* and *EFIJ*. The horizontal surface *LEH* is exposed in the finished dome, *HE* being the springing line of the outer spherical surface, as shown in plan at *he*, Fig. 205. One of the two voussoirs, equal in every respect, which contain the remaining part of the pendentive, is shown in Fig. 208,

its position relative to 207 being indicated. The lower bed WTH^1G^1 lies in the plane containing $MHKG$, Fig. 207, H^1G^1 coinciding with HG. The inner face $PNQG^1$ is spherical, QG^1 being an arc of the semicircular line of intersection between the spherical and vertical plane surfaces. The upper bed $PNXY$ is conical, being in the part of the dome above the pendentives; the nature of this surface has been fully described in a previous problem. The outer surface RWH^1XY is partly in the outer spherical surface and partly in the outer vertical plane surface, RH^1 being part of the line of intersection between these surfaces. The positions of the various surfaces in the dome are shown in elevation and sectional elevation, Fig. 205, the figures being lettered to correspond. Thus, RH^1, Fig. 208, is r^2h^2, Fig. 205; and PQT, Fig. 208, is $p^1q^1t^1$, Fig. 205.

The dotted lines, Fig. 208, indicate the block of stone from which the voussoir may be worked. The circular arris XY is in the upper surface of the block; the plane face RWH^1 is in one of the vertical surfaces; $RWYPQT$ is in one vertical end surface; the arris TG^1 is in the lower horizontal surface; the point N is in one of the vertical surfaces; the vertical joint plane XH^1G^1N is inclined to the vertical end surface of the block, only the point H^1 being in this surface.

To understand fully the conditions governing the shapes of the voussoirs, the student must work a similar example for himself.

Spherical niches involve no special geometrical operations outside those already discussed in connection with domes, excepting one or two comparatively simple problems that arise in niches that are less than a semicircle in both plan and elevation. An example of this type is shown in Fig. 209, in which half plan, half elevation, and vertical cross section are given. Let o be the centre of the plan curves, and o^1 the centre of the elevation curves, and let the centre lines of the ribs be required to lie in vertical planes through the centre of the sphere. The method of projecting the elevations of the ribs is shown fully in the figure and should not require any explanation. The curvature of the ribs is equal to the curvature of a great circle of the sphere, and is determined as follows: Draw h^2s in the same straight line with q^1m^1, and draw tp^2 perpendicular to it. Draw n^1n^2, nn^3, and ot parallel to h^2s, and with n^3 as centre, n^3t as radius, cut nn^3 produced in j. Draw a horizontal from o^1 to meet a perpendicular from j in O; then On^2 is the radius of the inner edges of the ribs. Project p^1 to p^2 and draw the outside curve, centre O, from p^2 to h^2. If the horizontal, vertical, and radiating ribs are each required to have one edge in the inner and

one in the outer spherical surface of the niche, rm must be made equal to h^2f^2 and p^1n^1 equal to p^2n^2.

The length of any radial rib may be found by turning the points in

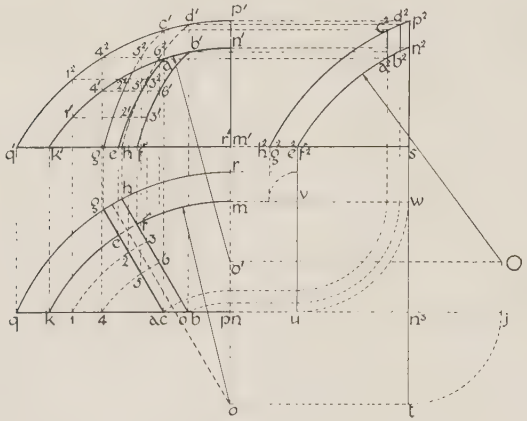


Fig. 209.*

the upper end about o into nr, projecting into the perpendicular vu, turning about v into vw and projecting into the section ; thus, $h^2f^2d^2b^2$ is the length of the rib shown in plan and elevation, and $a^2b^2c^2d^2$ is the elevation of the bevel on the upper end.

CHAPTER XIII

THE CYLINDER

It is assumed that the student is already acquainted with the more elementary properties of the cylinder.

Problem 98. *Fig.* 210.—**Given the projections of a right circular**

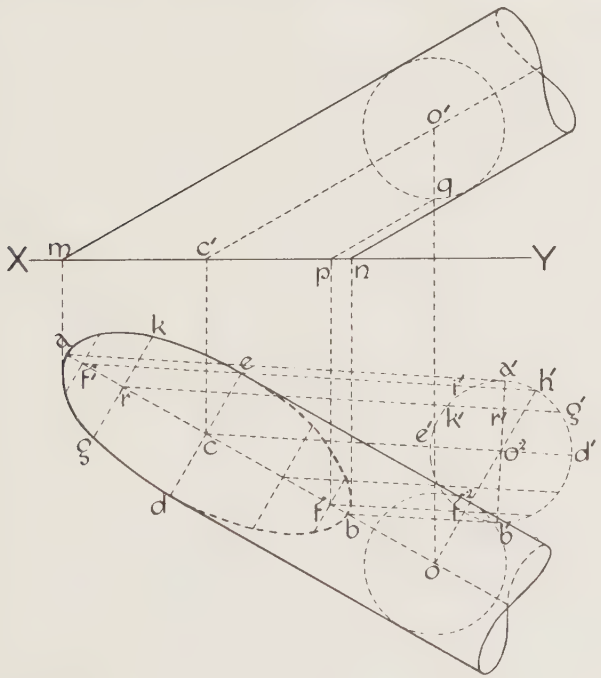


Fig. 210.*

cylinder inclined to both planes of projection, to determine a horizontal section of the cylinder.

Let oc, o^1c^1 be the projections of the axis, and let the circles, centres o and o^1, be the projections of a sphere inscribed in the cylinder.

Draw oo^2 perpendicular to co and equal to the height of O above the horizontal plane.

Join o^2c, and with o^2 as centre draw a supplementary projection of the sphere. Draw b^1a^1 perpendicular to co^2, and draw parallels to o^2c from b^1 and a^1 to meet oc in b and oc produced in a. Then ab is the major axis of the horizontal elliptic section, c is its centre, and de the minor axis: the section may now be determined by the trammel, or as a projection from the major and minor circles. Alternatively, draw any straight line, as k^1g^1, parallel to o^2c and produce to meet ab, and draw a perpendicular gk to ab through the point of intersection r. Make rk and rg each equal to r^1k^1 or r^1g^1; then g and k are points in the outline of the required section, and any number of points may be obtained in a similar manner. The focal points may be found by projecting f^2 and h^1 to f and f^1; then f and f^1 are the foci. Perpendiculars from m and n are tangents to the section outline.

NOTE.—The axis of the elliptic section may be found direct by drawing qp parallel to c^1o^1 and projecting p to f. Then since f is the focus, cb and ca are each equal to ef or df, and the section may be determined by any one of the several methods of constructing an ellipse, the axes being given.

Problem 99. *Fig.* 211.—**Given the projections of a right circular cylinder inclined to both planes of projection, to determine the section of the cylinder by any oblique cutting plane.**

Let cm and c^1m^1 be the projections of the axis, and let vth be the traces of the oblique section plane.

Draw the elevation of the inscribed sphere, centre o, to touch the ground line in f^1. Project f^1 to meet cm in f. Through c, the horizontal trace of the axis of the cylinder, draw kn perpendicular to cm, making ck and cn each equal to the radius of the cylinder. Make cg and ch each equal to nf; then nk and gh are the axes of the elliptical intersection between the cylinder and HP. Draw the major and minor circles of the ellipse and determine points in the curve by projections from these circles, as explained in the chapter on conics; thus, cq is any radius of the major circle, cutting the minor circle in r; then if rd is parallel to the major axis and qd is parallel to the minor axis, d is a point in the ellipse. Produce mc to meet xy in b and project b to b^1. From a, the point of intersection between cm and ht, draw a perpendicular to meet xy in a^1. Join a^1b^1. Project h to h^1 and from h^1 draw a parallel to c^1m^1 to meet a^1b^1 in 4^1. Project 4^1 to meet cm in 4; then 4 and 4^1 are the projections of a point in the section outline. Project d to d^1 and draw a straight line from d parallel to cm to meet ht in p. Project p to p^1 and from p^1 draw a parallel to a^1b^1 to meet

a parallel to c^1m from d^1 in 3^1. Project 3^1 to 3, in dp; then 3 and 3^1 are projections of a point in the section line. Any number of points in the elliptic section line may be found in the same way. For $h4$ may be regarded as the plan of a generator of the cylinder, h its horizontal trace, and h^14^1 its elevation. If a vertical plane is arranged to contain this generator it will intersect the given plane in a line of intersection whose projections are ab and a^1b^1. Since AB and $H4$ are in the same plane they intersect in the point 4; and since AB is in the plane vth and $H4$ is in the surface of the cylinder, the point of intersection 4 is

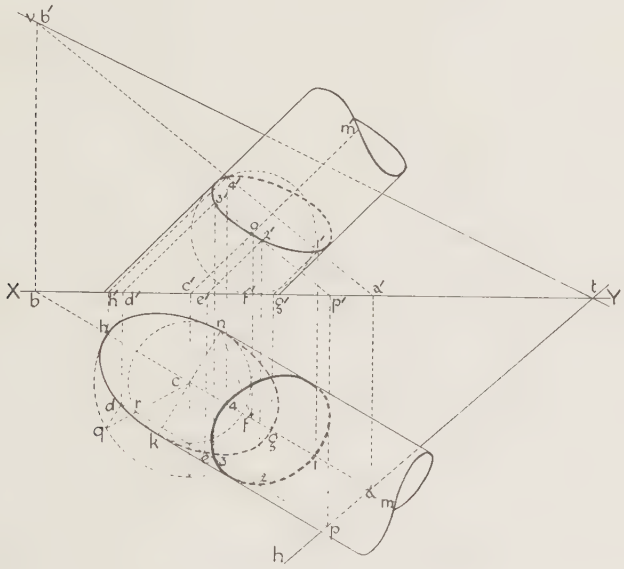


Fig. 211.

in the plane and in the surface of the cylinder, and is therefore a point in the line of intersection. The horizontal section of the cylinder is not required to be drawn in this solution, but is included in the figure to assist in making the construction clear.

Problem 100. *Fig.* 212.—**Given one projection of a point on the surface of a cylinder inclined to both planes of projection, to find the second projection.**

Let ab, a^1b^1 be the projections of the axis of the cylinder, and let p be the plan of a point on the surface of the cylinder.

Draw the circle, centre o, to touch the outline generators of the elevation of the cylinder and to touch the ground line at f^1. Project

f^1 to f on the plan of the axis; then f is the focus of the elliptic intersection between the cylinder and HP. From a, the horizontal trace of the axis, draw ac perpendicular to ab and equal to the radius of the cylinder; then ac is the semi-minor axis of the horizontal section. Make ad equal to cf, and with a as centre draw circular arcs through c and d. Through p draw a straight line parallel to ab to cut the arc of the minor circle in k. Draw the radius ak and produce to cut the arc of the major circle in h. From h draw a straight line to cut pk per-

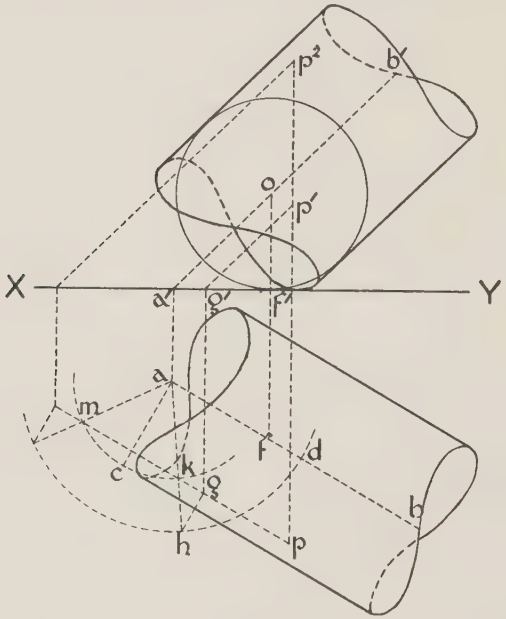


Fig. 212.

pendicularly in g; then g is the horizontal trace of the generator of the cylinder through the given point.

Project g to meet xy in g^1, and from g^1 draw a straight line parallel to a^1b^1 to meet a projector from p in p^1. Then p^1 is the required projection.

Note.—p^1g^1 is the elevation of a generator whose plan is pg. A second solution for a point p^2 on the upper surface of the cylinder may be found by producing pk to cut the minor circle in a second point m, from which the elevation of a second generator whose plan passes through p may be determined as shown.

Development of Lines on Cylindrical Surfaces

Fig. 213.—A portion of a right circular cylinder is shown in plan at ab and in elevation at $a^1b^1a^2b^2$.

The axis of the solid is vertical, the upper end is in an inclined plane, and the lower end is a curved surface having the quadrant of a circle a^1b^1 as its elevation. The development of the cylindrical surface lying between the plane and curved end surfaces is shown at $BB^1AA^1B^2B^3$. The development is obtained as follows: Divide the plan circle into any desired number of equal parts and regard each division point as the end view of a generator. Draw the elevations of the generators. Set out any straight line MN perpendicular to

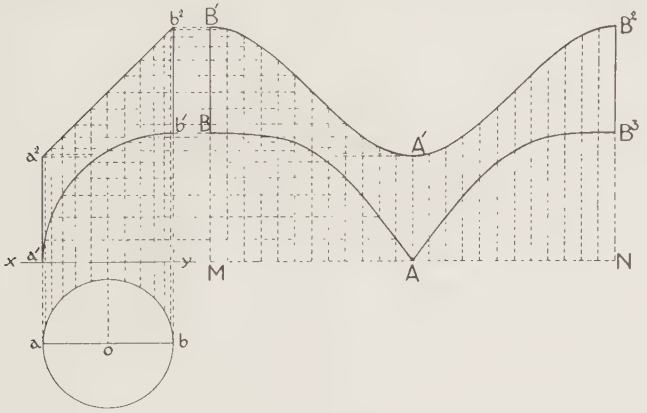


Fig. 213.

the axis and equal to the circumference of the cylinder, and divide it into the same number of equal parts as are contained in the plan circle. Erect perpendiculars to MN from the points of division and project the points of intersection between the elevations of the generators and the ends of the cylinder on to these perpendiculars, commencing with any desired generator, but taking them in sequence around the cylinder. In the figure the cylinder is shown opened along the generator BB, therefore MA equals half the circumference of the cylinder and AA^1 equals a^1a^2.

It will be seen that the development of any line in the cylindrical surface may be obtained in this manner.

Problem 101. *Fig.* 214.—**To develop the veneer for a wreathed open string in a quarter-turn geometrical stair, having given the positions and height of the risers.**

The quarter circle ab, centre o, is the plan of the cylindrical surface

of the string, and is tangent to the plans of the faces of the straight strings *fb* and *da*. The positions of the risers are shown in plan at 3, 4, 5 . . . 9 ; the widths of the treads 4 5, 5 6, 6 7, 7 8, between *a* and *b*, being equal in this example. The points *a* and *b* are the plans of the vertical lines of intersection between the cylindrical surface and the plane surfaces, and these lines of intersection are called the springing lines. To develop the face of the string, set out on *fb* produced the distance *bA* equal to the length of the arc *ab* and divide *bA* into four equal parts to correspond with the divisions along the arc. Make the distance $A3^1$ equal to the width of the parallel step 4 3. (The parallel steps are called fliers ; the tapering steps are called winders.) At any convenient position set up a straight line *MN* perpendicular to *bA*, and, starting at any desired point in this line, mark off a number of distances each equal to the given height of riser ; the number of distances should equal the number of steps required in the development, and must be numbered to correspond with the plan numbers. Draw a perpendicular from 3^1 to meet horizontals from *M* and 3^2 in d^1, and 3_1, and continue as shown to $f^1 9_1$.

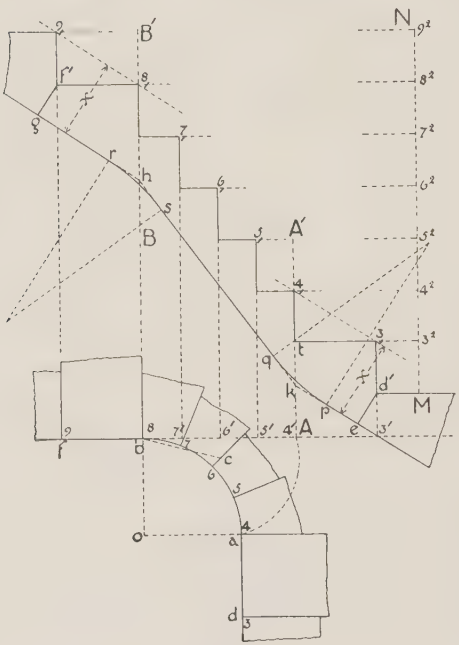


Fig. 214.

Draw the springing lines AA^1 and BB^1. Draw *gh* parallel to $9_1 8_1$, and *ke* parallel to $4_1 3_1$, making the distance *x* between these parallels equal to the required width of the straight string. Because $f^1 8_1$ and $i 3_1$ are each equal to the width of the fliers, *gh* and *ek* are inclined at the same angle as the straight strings of which they are a part.

Draw the straight line *hk* and draw short arcs *pq* and *rs*, tangent to *gh*, *hk*, and *ke* at *r*, *s*, *q*, and *p*. Draw gf^1 and ed^1 perpendicular to *gh* and *ek* respectively. The method of making the bent string is briefly as follows: A piece of wood, equal to the thickness of the straight strings, is set out to contain all the lines between gf^1 and $d^1 e$, including

the latter lines and the springing lines AA^1 and BB^1. The full width of the material lying between the springing lines is reduced to the thickness of a veneer, the recessing being on one side only and leaving the face of the veneer in the face of the string. A quarter cylinder of radius *oa* and length equal to *MN* (approx.) is built, around which the veneer may be bent. Narrow strips of wood called staves are then glued to the back of the veneer to restore its strength and thickness. When the glue is hard, the string is removed from the drum and cut to the lines already set out on the face of the veneer. The joints at gf^1 and ed^1 between the curved and straight portions of the string are then made.

To set out an open wreathed string in a half-turn geometrical stair having diminished fliers ; the widths of the treads along the string to be determined from the development (Fig. 215).

Draw the plans of the faces of the straight strings *Da* and Kc^1 and the semicircular turn abc^1. Draw straight lines to meet the straight strings perpendicularly in the points 3, 4, 5, and 14, 15, 16, to represent the faces of the risers of the parallel fliers. The positions of these lines are decided by the space allotted to the stairs, the number of steps required, and the perpendicular distance between the floors. In the present example it is assumed that the stair contains seventeen treads and eighteen risers. The distance around the string from 5 to 14 is to be filled in with nine graduated steps. The four steps, 5 6, 6 7, 12 13, 13 14, are called " balancing steps " or diminished fliers, and are sometimes referred to as " danced " steps.

In deciding the number of steps required in stairs containing winders, a line called the walking line is drawn parallel to and at a distance of from fifteen to eighteen inches from the string, and the steps are arranged to have an equal width along this line. The line *ZL* on the right of the figure is part of the storey rod, and is divided into as many equal distances as there are steps in the length of stair being considered, each distance being equal to the required height of riser. Aa^1 and Cc^1 are the developed springing lines. The horizontal distances between Aa^1 and *re*, and between Cc^1 and *ux*, are equal to *a*5 and *c*14 respectively. The steps below *f* and above *g* in development are parallel fliers and are projected from the storey rod so as to leave ten divisions 5^1, 6^1, 7^1, 8^1 . . . 14^1 as risers for the winders and diminished fliers. Straight lines through *f*, *d* and *g*, *k* are drawn to meet the springing lines in a^2 and c^2, and a straight line is drawn between the latter points. b^1 is the mid point of a^2c^2, and c^2m equals c^2b^1 and a^2p equals a^2b^1 ; therefore pa^2, a^2b^1, b^1c^2, and c^2m are all equal. The circular arcs $prstb^1$

and $muvwb^1$ intersect tangentially at b^1, the former is tangent to da^2 at p and the latter is tangent to kc^2 at m. The division points in the storey rod ZL are projected horizontally to meet the two tangential arcs in points r, s, t, w, etc., and the tread and riser lines are drawn to intersect in these points.

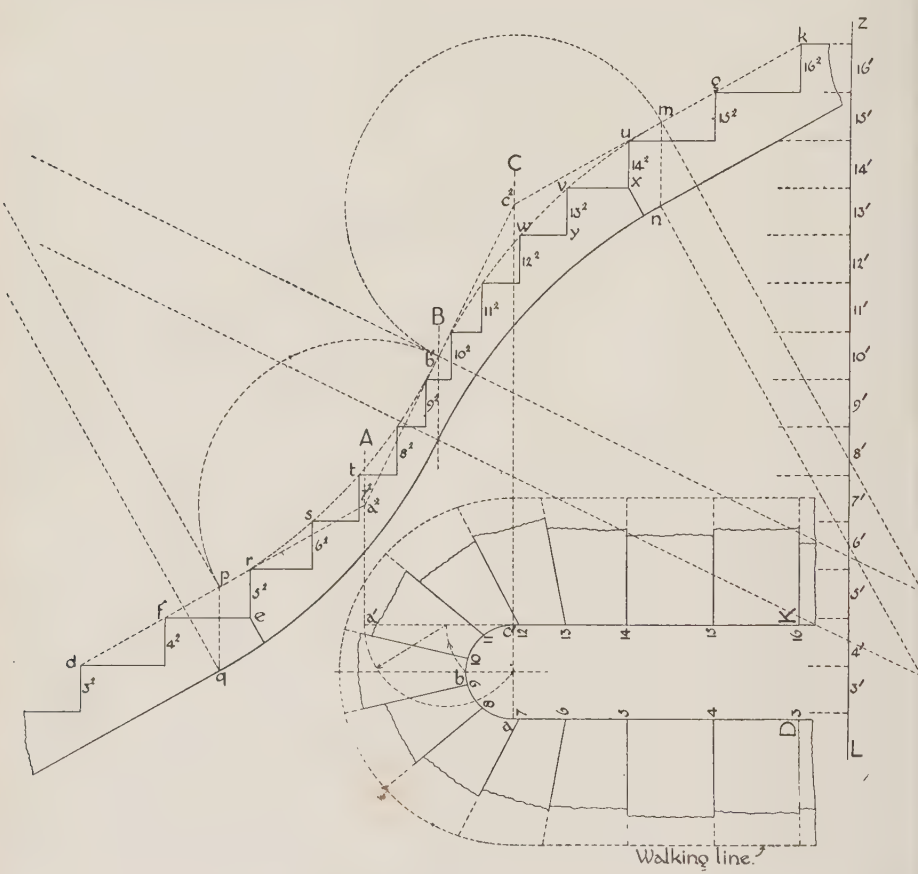


Fig. 215.*

The widths of the treads thus obtained are transferred to the plan, as shown at 5, 6 . . . 14, and the straight lines joining these points to the points of division along the walking line are the plans of the faces of the risers. The line qn representing the lower edge of the string is equal in every respect to the nosing line pb^1m, and is obtained by drawing straight lines parallel to dp and mk to represent the required widths of the straight sections of the string, to intersect perpendiculars

PLATE V.

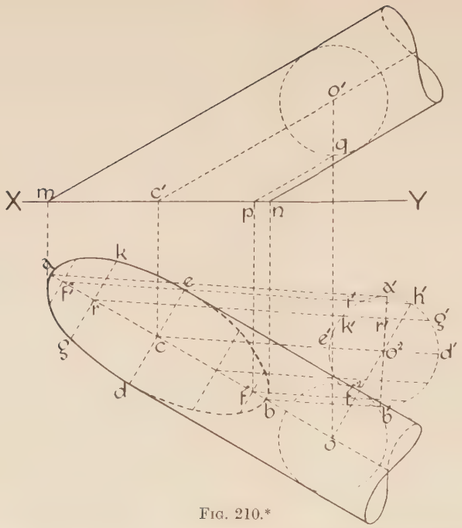


FIG. 210.*

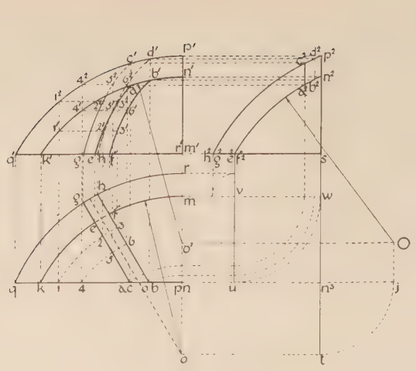


FIG. 209.*

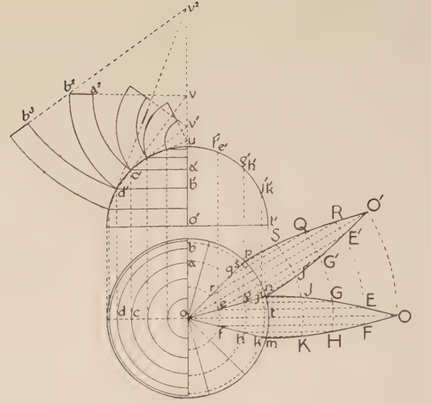
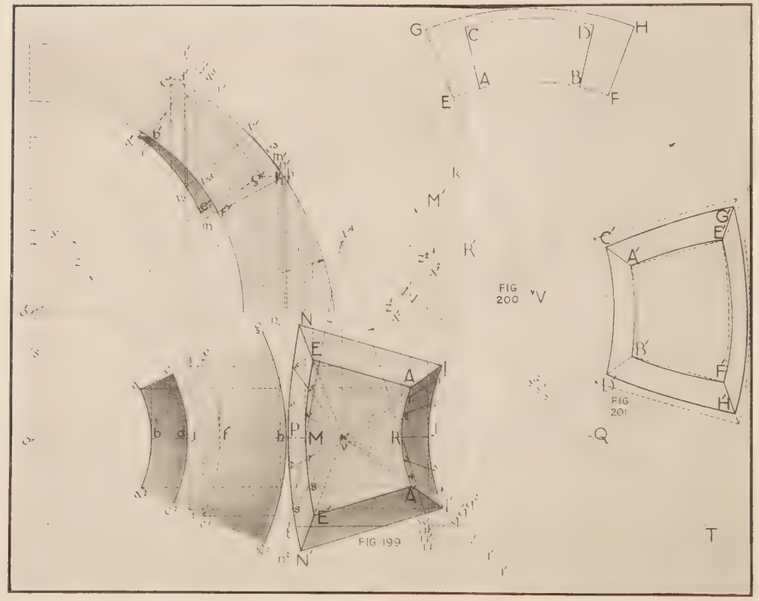


FIG. 197.*



FIGS. 199, 200, and 201.*

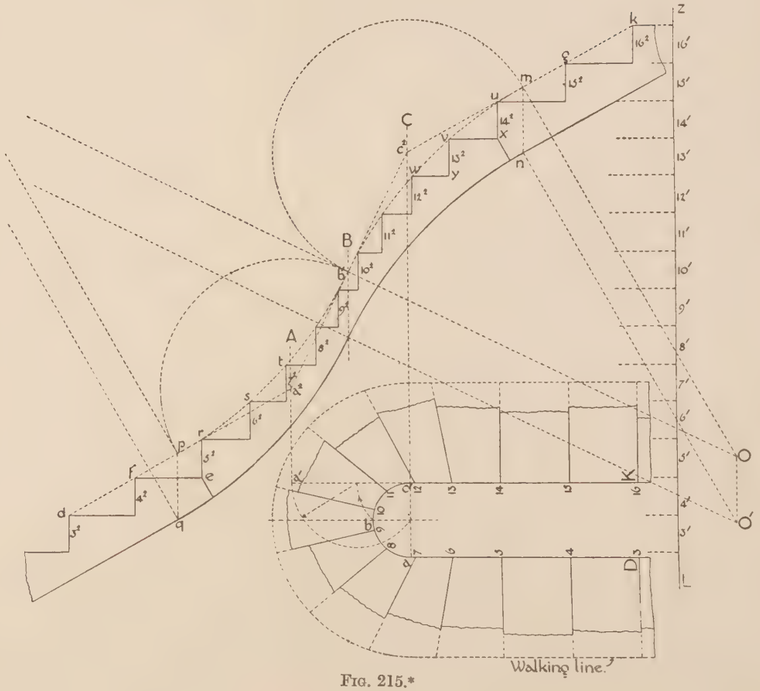


FIG. 215.*

Face page 200.

from p and m in q and n, and making OO^1 parallel and equal to mn. Then O^1 is the centre of the upper curve, tangent to the straight string at n, and a similar point for the centre of the lower curve is situated on the other side of the development, q being the point at which curve and straight intersect tangentially. The difference between the mean widths of the string at the centre line Bb and the portions above and below the springing lines is only apparent before the string is bent; when made cylindrical a constant vertical height gives the desired parallel effect. The construction of the wreathed string is carried out in a manner similar to that briefly described in the previous example.

The Helix.

If a rectangle revolve at a uniform rate about one of its sides, which remains fixed, any point moving at a uniform rate in one direction along the opposite side will generate a curve known as a helix or helical curve.

If the surface of the cylinder generated by the revolving rectangle be developed, the development of the locus of the moving point will be a straight line.

The distance through which the point moves along the side of the rectangle while the rectangle is making one complete revolution is called the axial pitch of the helix; the perpendicular distance between two helices of equal axial pitch on the surface of the same cylinder is called the normal pitch.

Problem 102. *Fig.* 216.—**To draw the projections and development of a helix of given radius and given pitch.**

With the given radius draw the circle, centre O, and divide the circumference into any desired number of equal parts, numbering the points of division 1, 2, 3 . . . 12. Regard this circle as the plan of an upright circular cylinder, and the division points as the plans of a series of generators of the cylinder. Draw the elevation of the cylinder, making its length equal to the given pitch, and draw equidistant horizontal lines to divide the elevation into the same number of parts as there are in the plan circle. Commencing at any desired division point in plan and taking the generators in order, draw projectors to meet the elevation horizontals in successive order to obtain the points 1^1, 2^1, 3^1 . . . 12^1. A smooth curve through these points, tangent to the outline generators of the cylinder, is the elevation of the helix.

Produce b^1a^1 to A, making a^1A equal to the circumference of the plan circle. Divide a^1A into as many equal parts as are set out on the plan circle and draw perpendiculars to meet the elevation horizontals produced in points 1^2, 2^2, 3^2 . . . 12^2. The line through these points

is a straight line and is the development of the helix. By dotting the upper half of the curve, as in the figure, the helix becomes left-handed ; if the lower half were dotted and the upper half firm, the helix would be right-handed.

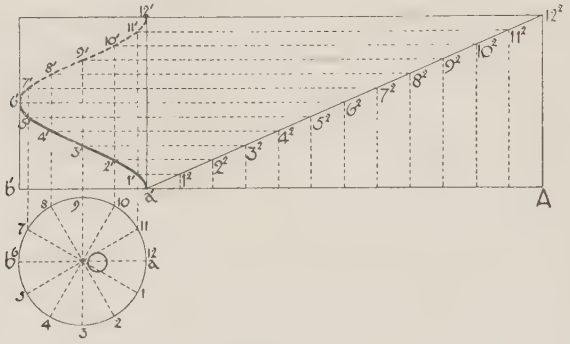


Fig. 216.*

Helical Surface.

Fig. 217.—If one of two intersecting straight lines revolves about the second line, which remains fixed, as an axis, the angle between the lines remaining constant, and also moves along the axial line, the rates of the two movements having a constant ratio, the surface generated by the moving line is called a helical surface. Every point in the generating line, other than the point of intersection with the axis, traces out a helix, and all the helices in the same helical surface are of equal pitch.

In Fig. 217, a is the plan and aa^1 the elevation of a fixed straight line, about which a horizontal straight line has moved so as to generate a helical surface, and seven positions of the generating line are shown in the figure.

Helical surfaces, also known as screw surfaces, occur frequently in building and engineering in connection with geometric stairs and handrails, skew arches, screw threads, etc. The surface is undevelopable, and the only straight lines that may lie in it are those coinciding with positions of the generating line.

Square screw threads, and the more usual form of wreathed handrails, are sections of right cylinders, the sections being each contained between two parallel helical surfaces and two cylindrical surfaces. Referring to Fig. 218, in which the projections of a section of this type are given, the nature of the solid will be best understood if it is considered as having been generated by a plane rectangle moving so as to have

its plane always vertical and normal to the surface of the cylinder whose axis is the vertical line, plan a, elevation a^1a^2, and one of its upright

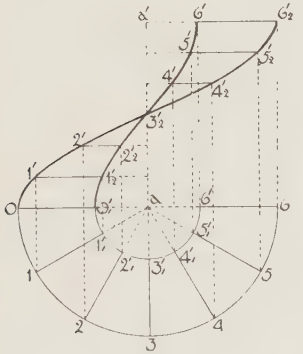


Fig. 217.

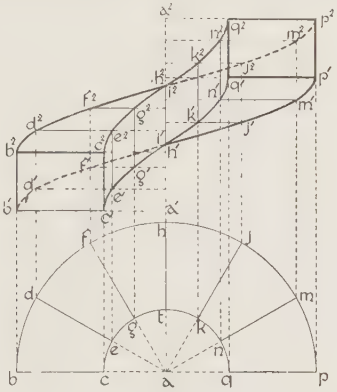


Fig. 218.

sides always in the surface of the cylinder. Seven positions of the rectangle are shown in plan and elevation at bc, $b^1b^2c^1c^2$ . . . pq, $p^1p^2q^1q^2$. It will be seen that each angular point of the rectangle generates a helix, the four helices being of equal pitch. The horizontal sides of the rectangle generate helical surfaces, the two surfaces being equal in every respect. The two upright sides generate cylindrical surfaces, the plans of which are the two semicircles *cekq* and *bfjp*.

From the foregoing it will be seen that any section of the solid by a vertical plane containing the axis of the cylinder is a rectangle; *no other plane section is rectilinear*. Consider Fig. 219, in which ab and a^1b^1 are the projections of the centre line helix in a solid similar to the one explained above. AB is the straight line development of the centre line and SS^1 is the edge view of a cutting plane perpendicular to AB. hh^1, ii^1, jj^1, and kk^1 are

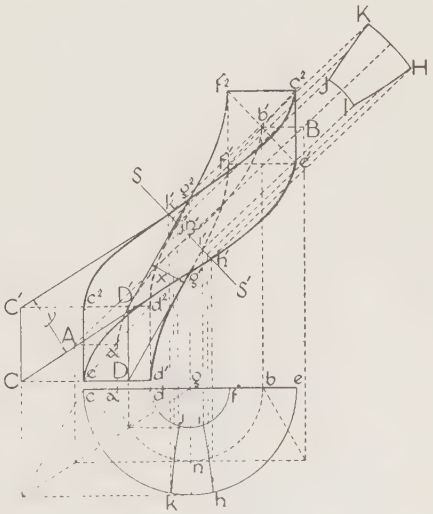


Fig. 219.*

the projections of the points in which the plane cuts the helical arrises. $HIJK$ is the true shape of the section, being a projection on to a plane parallel to SS^1. Since the cylindrical surfaces are cut by a plane inclined to the axis of the cylinder, the curves KH and JI are elliptical; and since the lines of intersection between the cutting plane and the helical surfaces do not coincide with positions of the generating straight line, JK and HI are curves. The nature of these curves will be discussed later.

Another feature of the solid, of special importance to students of handrailing, is the difference in the normal pitches of the two inner and the two outer helical arrises. Referring to the figure, Dg^1 and D^1g^2 are the straight line developments of the lower halves of the two inner helices, and the mean distance between them is indicated at x. Cg^1 and C^1g^2 are similar developments of the outer helices, and y is the mean distance between them. It is important to note that x is considerably less than y, and that the difference varies as the radii of the cylindrical surfaces vary. This condition frequently obliges the practical handrailer to distort the true geometric section of wreathed handrails so as to make the process of moulding them practicable.

Sections of Helical Surfaces.

Fig. 220.—The straight line ABC is the edge view of a plane which cuts the helical surface shown in elevation through 1^1, 2^1, 3^1. The projections of twenty-four positions of the generating line are given, and the points of intersection between ABC and the elevations of the generators are projected to cut the plans of the corresponding generators, produced where necessary in points $a, b, c \ldots t$.

The section has two branches, $abc \ldots j$ and $klm \ldots t$; and if the generating line be regarded as extending indefinitely in both directions, each branch is a curve of indefinite length moving away from the axis and approaching two straight lines, one on each side of curve, asymptotically; one of the straight line asymptotes being common to the two branches.

Consider the straight lines AA^1, BB^1, and CC^1. The generators $O1$ in plan, 1^1 in elevation, $O2$ in plan, 2^1 in elevation, and $O3$ in plan, 3^1 in elevation, are each parallel to the section plane, and therefore will not meet the plane, however far produced. All other generators are inclined to the section plane and will meet it if sufficiently far produced. It will be seen, then, that as the generating line approaches the position where it is parallel to the plane, its point of intersection with the plane becomes more remote from the axis, until, in the limit, the distance

becomes infinite, and the lines of intersection between the section plane and the planes that contain the parallel generators, and are perpen-

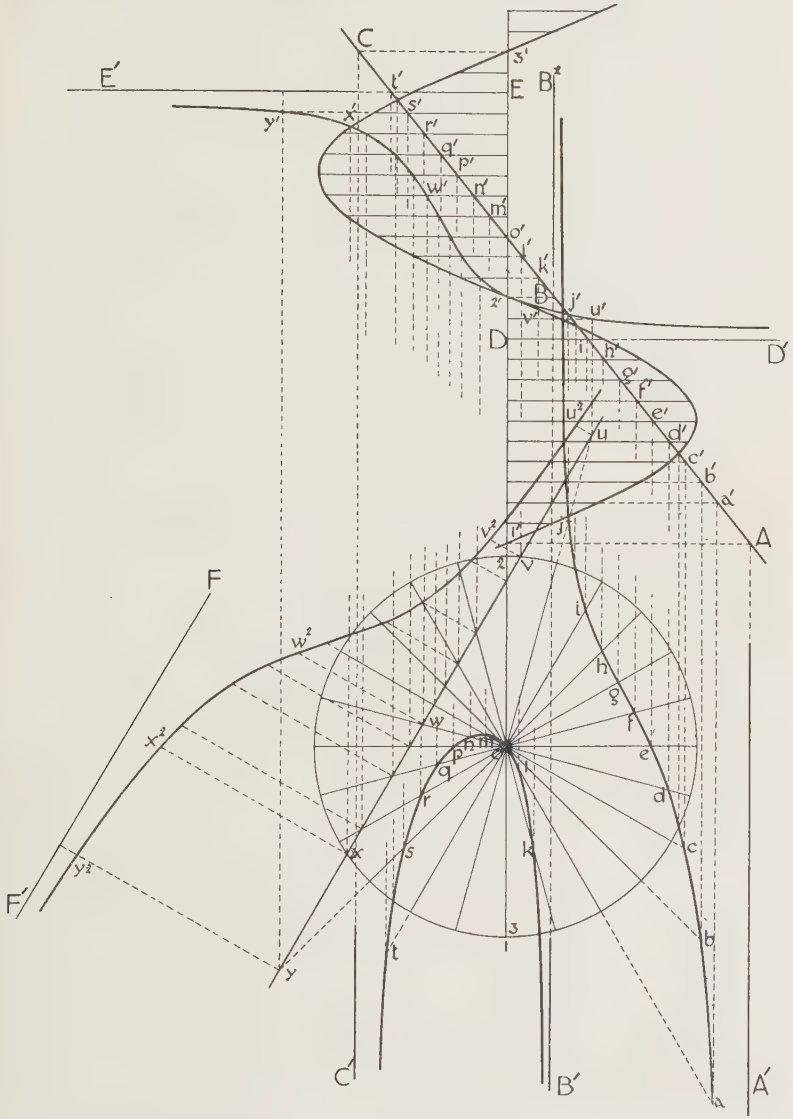


FIG. 220.

dicular to the axis of the helical surface, are asymptotic to the section line.

The true shape of the section may be obtained by a projection perpendicular to ABC.

Now let the surface be cut by a plane parallel to the axis, as shown in edge view at $uvwxy$. The elevation of the section, $u^1v^1w^1x^1y^1$, is obtained by direct projection as before. If the surface extends indefinitely, the section is a curve of indefinite length, and the elevations of the two generators that are parallel to the section plane are asymptotes to the elevation of the curve. Thus, if ot and oi are the plans of these generators, EE^1, through t^1, and DD^1, through i^1, both straight lines perpendicular to the axis, are asymptotes to the curve $u^1v^1w^1x^1y^1$.

The true shape of this section line is shown at $u^2v^2w^2x^2y^2$, the heights of the points above uy being equal to the perpendicular distances of the corresponding points from DD^1. The plan of the section plane uy, and the straight line FF^1 parallel to it, are asymptotes of the curve u^2y^2, the perpendicular distance between uy and FF^1 being equal to the perpendicular distance between DD^1 and EE^1.

Normals and Tangents to Plane Sections of Helical Surfaces.

Fig. 221.—Let egb and efd be the plans of two helices generated on the surfaces of the hollow cylinder whose end section is $a^1b^1c^1d^1$ (only one quarter of the cylinder is shown) and let the helical surface between the helices be cut by the section plane SS. Then $f^1h^1g^1$ is the elevation of the section and $f^2h^2g^2$ is the true shape of the section. The section curve is extended through i to o, and in the other direction through j to show its nature more clearly, and is obtained in the manner explained in connection with the preceding figure.

Let ep and eq be the straight line developments of the two helices (op and oq being equal to the developed lengths of the two quadrants, a^1b^1 and c^1d^1, respectively) and b^1s and b^1r be respectively equal to the inner and outer radii of the cylinder. Then, if sk^1 is drawn parallel to eq and ok is made equal to b^1k^1, k is in the tangent to the elevation of the section curve at its point of intersection with the inner surface of the cylinder (f^1); and if o^2k^2 is equal to b^1k^1, a tangent to the true shape of the curve at the point f^2 will pass through k^2. NOTE.—$a^2b^2c^2d^2$ is the true shape of the section of the quarter cylinder by the plane SS, and f^2 is the point of intersection between the section line $o^2h^2j^2$ and the elliptic curve c^2d^2. Also if rn^1 is parallel to ep, and on is made equal to b^1n^1, a tangent to the elevation of the section line, $f^1h^1g^1$, will have its point of contact at g^1, the point in which the quadrant a^1b^1 cuts the section curve. Further, a straight line from s parallel to ep will intersect

a straight line from r parallel to eq in a point m^1, such that m^1 will be in the straight line joining k^1 to n^1, and if om be made equal to b^1m^1,

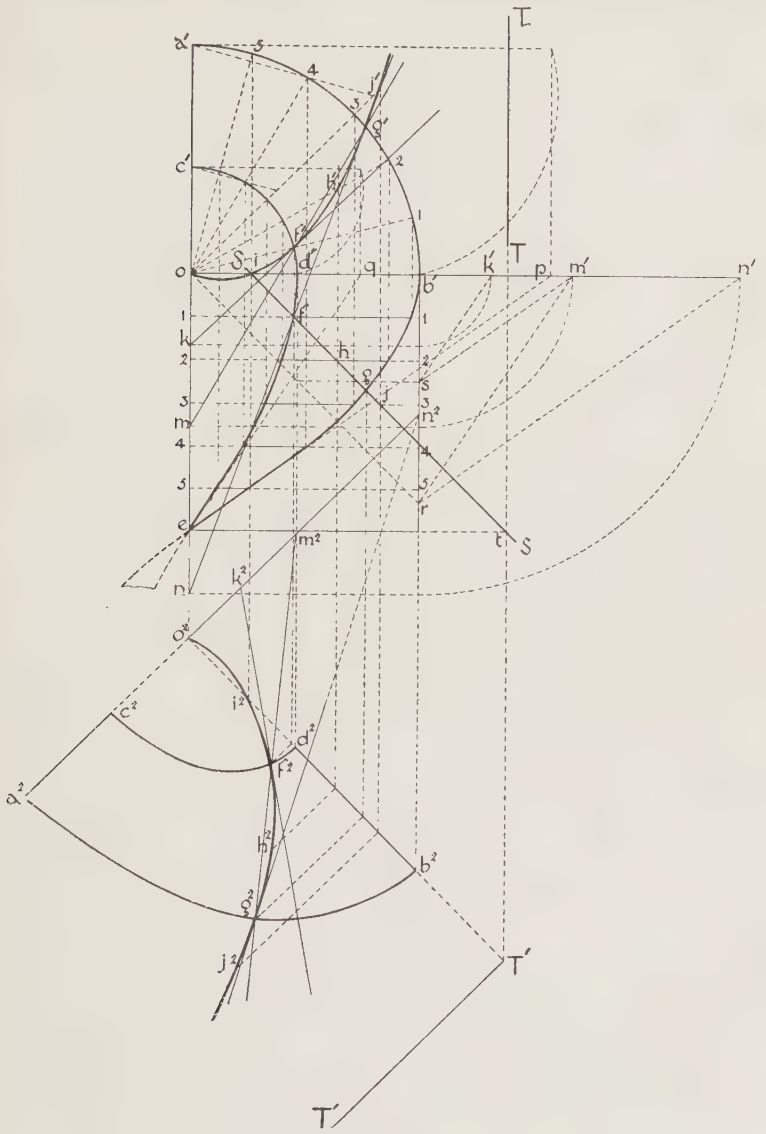


Fig. 221.

the chord f^1g^1 of the elevation curve will, if produced, pass through m. Again a tangent to the true section curve at g^2 will pass through n^2,

and the chord g^2f^2 will pass through m^2, where o^2n^2 equals on and o^2m^2 equals om.

Summarising: where the distance ok is derived from the inner radius and the inner helix, the tangent from k has its point of contact on the inner surface of the cylinder; and where on is derived from the outer radius and the outer helix, the tangent from n has its point of contact on the outer surface of the cylinder; where om is derived from the outer radius and the inner helix, the points of contact of the two tangents and the point m are collinear.

TT is asymptotic to the elevation of the curve oh^1j^1, and T^1T^1 is asymptotic to the true section curve $o^2h^2j^2$, these asymptotes being obtained by drawing a perpendicular to the axis of the cylinder from e to cut the section plane SS in t, and drawing TT through t parallel to the axis.

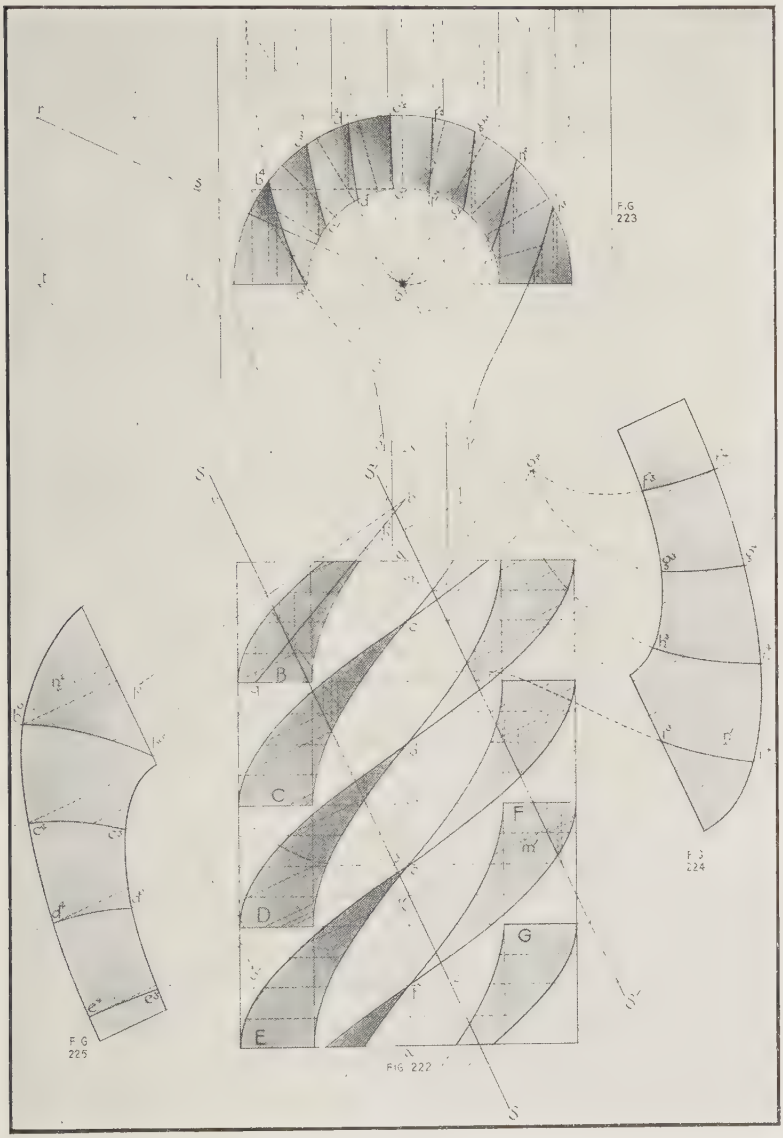
Sections of a Series of Parallel Helical Surfaces by a Common Plane.

Figs. 222, 223, 224, *and* 225.—In Fig. 222 the plans of a series of parallel helical surfaces B, C, D, E, etc. are shown. All the surfaces are cut by a plane inclined to the axis of the cylinder, the plane being arranged to cut each of the surfaces at a position other than the point of intersection between the plans of the outline helices. The plane is shown in two sections SS and S^1S^1, the positions of these sections being such as to give precisely the same result as would be obtained from one continuous plane across the full range of sections; thus SS intersects the surface F at a position corresponding to the intersection between S^1S^1 and the surface C. By this arrangement the sections of eight equal and equidistant helical surfaces by a common plane may be obtained with greater compactness.

In Fig. 223 the eight section lines are projected on to a plane perpendicular to the axis of the cylinder whose surfaces contain the outline helices, the parts of the section lines contained within the thickness of the cylinder being shown in full lines, the continuations of the curves being dotted. The section of surface B is the tortuous curve $b^1b^2b^3b^4$ asymptotic to straight lines from v and y parallel to the axis, v and y being the points in which perpendiculars to the axis from b and e intersect the section plane.

The sections of the surfaces C, D, E, F, G, and H (H is not shown in the figure, its section being identical with the section of E by the plane S^1S^1) are the curves oc^1c^2, od^1d^2, oe^1e^2, of^1f^2, og^1g^2, and oh^1h^2, these curves being similar in character but of varying degrees of curvature. Each curve has a straight line asymptote which is determined by pro-

jecting the point of intersection between the plans of the outline helices at right angles to the axis on to the section plane, as previously explained.



FIGS. 222, 223, 224, and 225.*

An eighth helical surface I (not within the figure) would, if cut by the plane SS produced, have a section line identical with the section $i^1 i^2 i^3$

of the surface F, this section being similar to b^1b^4, but of less curvature. The true shapes of the section curves through the surfaces F, G, H, and I are shown in Fig. 224, the quarter ellipses f^3, g^3, h^3 and f^4, g^4, h^4, i^4 being sections of the cylindrical surfaces by the plane S^1S^1. The true shapes of the remaining curves are shown in Fig. 225 across the section of the quarter cylinder by the plane SS. If Figs. 224 and 225 were brought together so as to have o^1 and o^2 coincident, and o^2i^3 in the same straight line with o^1b^5, they would form the complete section of the half cylinder above the axis produced by the plane SS.

Referring to Fig. 223, let the angle tor be equal to the angle between SS and aa (Fig. 222) and let rt and su be perpendicular to ot and equal to the outer and inner radii of the cylinder respectively. Let bp and bq be the straight line developments of the outline helices of the surface B; then the developments of all the outside helices will be parallel to bp, and the developments of the inside helices will be parallel to bq. Draw perpendiculars to bq from t and u to intersect ao in points m and k respectively, and perpendiculars to bp from t and u to intersect ao in m and n, then tangents to the elevations of the section lines at their points of intersection with the inner surface of the cylinder would all pass through k; thus, straight lines through kb^2, kb^3, kc^1, kd^1, ke^1, kf^1, kg^1, and kh^1 would be tangents to the curves. Since $i^1i^2i^3$ does not intersect the inner surface of the cylinder, there is no tangent to this curve from k. Tangents to the elevation curves at their points of intersection with the outside cylindrical surface would all pass through n; thus straight lines through nb^1, nb^4, nc^2, nd^2, ne^2, nf^2, ng^2, nh^2, ni^3, and ni^1 would all be tangents. The point m is in the chords (produced) through b^1b^2, b^3b^4, c^1c^2, d^1d^2, e^1e^2, f^1f^2, g^1g^2, and h^1h^2.

If the distances ok, om, and on be transferred to o^1k^1, o^1m^1, o^1n^1, and to o^2k^2, o^2m^2, and o^2n^2, the tangents and chords about the true shapes of the curves will, if produced, pass through the points in a manner to correspond with the elevation; thus k^1 is in the tangent to the curve $o^1d^3d^4$, the point of contact being d^3; and similarly for the other curves and points.

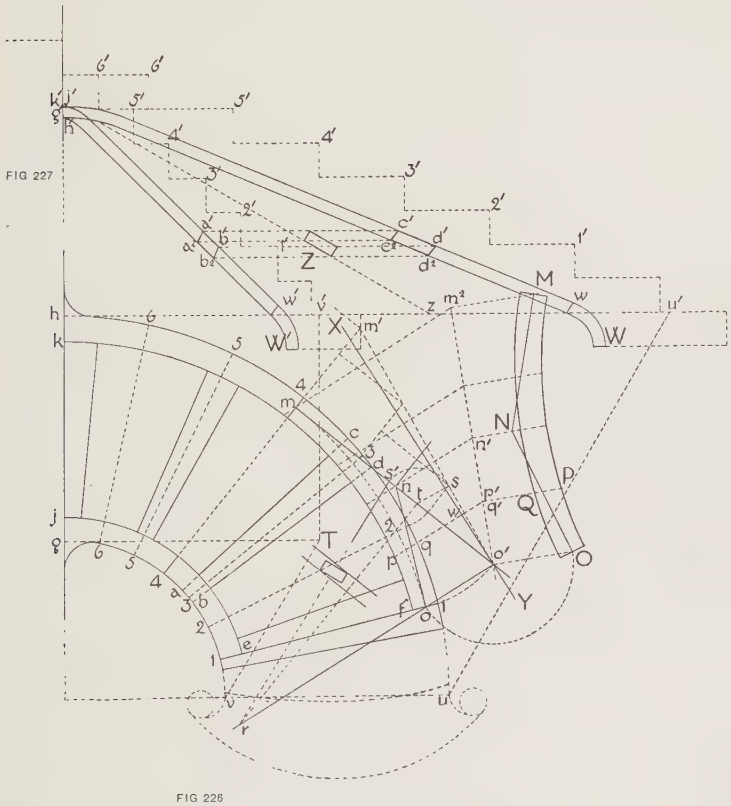
Helical Framing for Geometrical Stairs.

Figs. 226, 227, 228, *and* 229.—Fig. 226 is the plan of a panelled soffit framing under a quarter-turn geometrical stair, the positions of the straight risers being shown by the radiating lines 1 1, 2 2 . . . 6 6. There are two curved risers below 1 1, but these are not involved in the setting out of the framing.

The circular arcs gv and hu may be regarded as the plans of the centre lines of the strings, the latter being rebated an amount equal

to half their thickness to receive the framing. The widths of the curved styles are shown at *jg* and *hk*, the former being 4 inches and the latter 5 inches. The four rails are equal in every respect and have their arrises radiating ; this gives them each a taper from 6 inches to 3 inches.

Consider the rail shown at *abcd*. Since the inner end must intersect with the helical arrises of the inner style, and the outer end with the



Figs. 226 and 227.*

helical arrises of the outer style, and since these helices are of equal pitch but different radii, the faces of the rail will be twisted or winding surfaces. Theoretically the faces would be helical surfaces, but in practice the ends of the rail would be made rectilinear as shown in Fig. 228, the slight curves being produced in the process of cleaning up. To obtain the end sections of the rail, and their positions in the plank or board from which the rail is to be made, the straight line developments of the inner and outer edges of the framing must be determined

as shown in Fig. 227, where hv^1 and hu^1 equal the lengths of the quadrants gv and hu respectively. The developments of the steps along these quadrants are shown dotted at 1^11^1, 2^12^1 . . . 6^16^1, and the required width of the string gives the positions of the developed edges of the framing, which are shown finishing into the floor at W, W^1, and into the ceiling at g^1, h^1 with helical coves; the lower cove wW, w^1W^1 is worked separately out of the solid, the upper one being formed, if desired, by making the top rail of the framing sufficiently thick to fork across the styles and be worked to the required curve. The ends of the rail ab and cd, Fig. 226, are shown in development at a^1b^1, c^1d^1, Fig. 227, the centre cross section of the rail being made rectangular, as at Z; Zz being the developed centre line helix of the stair soffit. The corresponding angular points of the two end sections are of equal height, as a^1, c^1, since the four arrises of the rail are horizontal lines. In Fig. 228 enlarged views of the finished rail are shown and the end sections are lettered to correspond with the smaller sections in Fig. 227. Thus ab, Fig. 228, is parallel and equal to a^1b^1, Fig. 227; cd, Fig. 228, is parallel and equal to c^1d^1, Fig. 227; aa^1 and cc^1, Fig. 228, are parallel to a^1a^2 and c^1c^2, Fig. 227, respectively; and the four straight lines ca, c^1a^1, bd, and b^1d^1, Fig. 228, are all horizontal. A view of the rail on a plane parallel to the face of the material from which the rail is to be worked is shown in Fig. 228.

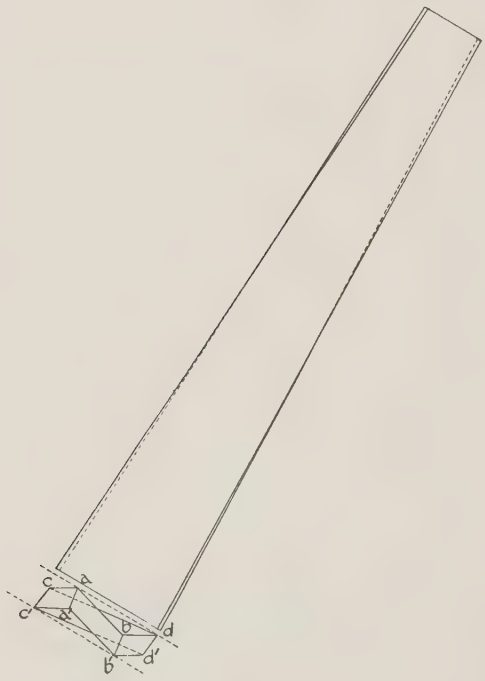


Fig. 228.

Each of the two curved styles is made in two sections, and may be treated as simple cases of wreathed handrails; the method of obtaining the face mould and bevel is shown in Fig. 226, and is as follows: Draw mn and on tangent to the plan of the centre line at m and o, and produce

mn to o^1, making no^1 equal no. Draw mm^1 at right angles to mo^1 and equal to the total height of the three risers 2 2, 3 3, and 4 4. Join m^1o^1, and from o^1 draw a straight line through o. Draw XY through o^1 perpendicular to o^1o. Erect a perpendicular to XY from m and on it mark the point m^2 distant from XY an amount equal to mm^1. Join m^2o^1; then o^1o is the horizontal trace and m^2o^1 is an edge view of the plane containing the tangent lines mn and on. Now suppose the rail to be part of a vertical hollow cylinder, the plan of the rail being part plan of the cylinder; then a section of this part of the cylinder, by the plane containing the tangents, will be the required face mould.

To determine the section any number of level ordinates may be drawn across the plan of the rail, parallel to o^1o, to meet m^2o^1, and the distances along these ordinates set out at right angles to m^2o^1; for example, let pp^1 be any straight line parallel to oo^1 and let p^1P be perpendicular to m^2o^1; then if p^1P is equal to wp and q^1Q is equal to wq, w being the point of intersection between the level ordinate and XY, P and Q are points in the true shape of the required section. Determine M, N, and O by setting the distances of m, n, and o from XY perpendicularly, out from m^2o^1, and draw the joint lines through M and O at right angles to MN and ON respectively. The outline curves of the mould are, of course, elliptic arcs, and may be determined by trammel or other suitable method.

The required twist bevel is the dihedral angle between the plane containing the two tangents mn, on and a vertical plane containing either tangent, and is obtained in the figure as follows: From any point t in mo^1 draw a perpendicular ts to m^1o^1, and a perpendicular to mo^1 from t to meet o^1o produced in r. With t as centre, and ts as radius, cut m^1o^1 in s^1 and join s^1r. Then the angle o^1s^1r is the required twist bevel.

The thickness of the material required is determined as shown at T, where the section of the finished style is drawn with its shorter sides (equal to the thickness of the rail) parallel to and equidistant from s^1r, and parallels to mo^1 are drawn to contain the section. The distance between these parallels is the required thickness.

Fig. 229.

The rail lying in the plank after the joints and edges have been worked is shown in Fig. 229.

A detailed explanation of the method of applying the bevel and mould will be understood after considering the section of the book dealing with geometrical handrailing.

Geometry of the Skew Arch.

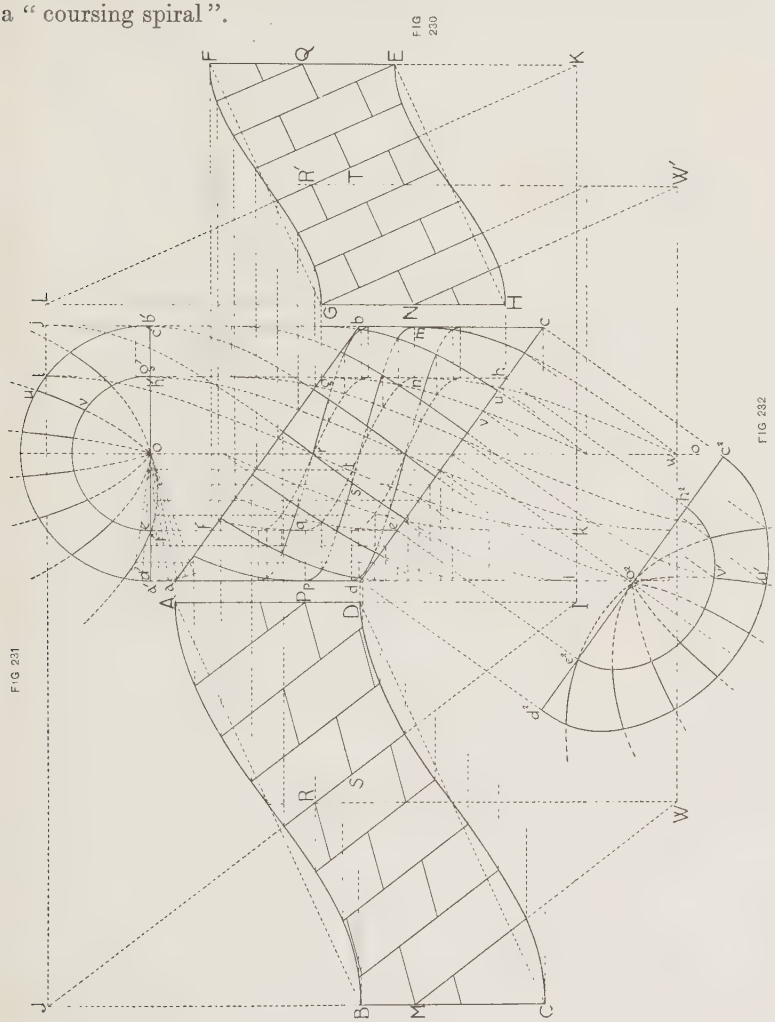
Fig. 230.—When the thickness of a skew arch is small enough to allow each voussoir to pass completely through from face to face, the joint surfaces may be plane, and the geometry involved is of a comparatively simple nature. In large skew bridges, however, where several voussoirs are necessary to obtain the required thickness, the question of stability introduces some extremely interesting problems. In ordinary arches the bed joints are plane surfaces normal to the arch soffit; in the skew arch this arrangement is not practicable, since it would result in having a section on each face of the arch with an abutment on one side only. To avoid this defect and at the same time retain a constant shape in the voussoirs, all the joint surfaces in the skew arch are made helicoidal. From a purely geometrical point of view—that is, ignoring any modifications necessitated by the question of expense or practical expediency—the problem is that of dividing a hollow cylinder into a number of parts, equal in size and shape, by two ranges of helicoidal surfaces. Fig. 230 may not be regarded as the setting out for a skew arch, but is designed to show in a somewhat exaggerated form the geometrical principles involved in this type of arch.

Referring to Fig. 230, let *abcdefgh* be the plan, and $a^1b^1 \ldots h^1$, Fig. 231, the elevation of part of a right cylinder lying between two vertical planes on *ab* and *cd*, the axis of the cylinder being in *HP*. *EFGH* is the development of inner cylindrical surface, *ABCD* a development of the outer surface. The developed inner face is divided into sections by straight lines drawn parallel and perpendicular to *HE* and *GF*; the parallels, as *NQ*, being omitted between alternate pairs of perpendiculars to suggest voussoirs. It has already been explained that a helix develops into a straight line; then all the straight lines parallel and perpendicular to *HE* will, when brought into the inner surface of the cylinder, be helices. Thus *nq* is the helix corresponding to the developed line *NQ* and *lk* is the helix corresponding to the developed line *LK*, the latter line and helix being continued to the extent of one half of the circumference of the cylinder; therefore *l*, *k*, *n*, and *q* are all in *HP*.

If now a straight line always normal to the surface of the cylinder and always intersecting the helix *nq* moves from the position *nm* to the position *qp*, it will generate a second helix *pm* on the outer cylindrical surface, and the helicoidal surface between the two helices would, in a skew arch, be called a "heading spiral".

Let *lj* be a second straight line moving so as to be always normal to

the cylindrical surfaces, and always intersecting the helix *lk*. In moving from *lj* to the position *ik* it will generate the helicoidal surface between the helices *lk* and *ji*, and this surface would, in the skew arch, be called a " coursing spiral ".



FIGS. 230, 231, and 232.*

Since the generating lines of the two helicoidal surfaces are always normal to the surfaces of the cylinder, they are always perpendicular to the axis ; and because the axis is horizontal, the plans of all the positions of the generating lines are at right angles to the plan of the

axis ; that is, the plan of any straight line perpendicular to the axis across either helicoidal surface is the plan of one position of the generating line. Then the line *st* joining the point of intersection of the outer helices to the point of intersection of the inner helices is the plan of the line of intersection between the two helicoidal surfaces, the line of intersection being a straight line normal to the cylindrical surfaces.

The development of the outer helix *pm* is shown, with sections omitted to correspond with *NQ*, at *PM*, and the developed outer helix *ij* is shown at *JI*, *S* and *T* being the developments of the points of intersection *s* and *t* respectively.

When all the straight lines in the two developed cylindrical surfaces are brought into the surfaces of the cylinder, they become the bounding lines of two ranges of parallel helicoidal surfaces, the two ranges cutting each other in normal, straight-line intersections.

The point of intersection between the plans of the inner and outer helices of each surface intersect on the plan of the axis, as at *r* and *w* ; *R*, R^1 and *W*, W^1 being the developments of the extremities of the generating lines at these positions.

All the complete rectangles on the developed inner cylindrical surface are equal and have corresponding sides parallel. All the complete rhomboids on the developed outer surface are equal with corresponding sides parallel. When the rectangles and rhomboids are brought into the surfaces of the cylinder, and each angular point in the rectangles is joined by a straight line, normal to the surface of the cylinder, to an angular point of a rhomboid, all the sections into which the cylinder has been cut by the two ranges of helicoids are equal ; each section having two cylindrical faces and four helicoidal faces.

It has been shown that all sections of a helicoidal surface by a plane not containing a generator are curved lines ; and since the range of surfaces corresponding to coursing spirals are cut by one or both the vertical end planes of the cylinder, the planes being inclined to the axis of the cylinder, the joint lines between the sections across the end planes are all curves similar to those explained in connection with Figs. 220 to 224. An elevation of the curves is shown in Fig. 231, and a second elevation, showing the true shapes of the curves and the elliptic end section of the cylinder, is given in Fig. 232.

In large skew arches, where the depth of face is small compared to the radius, the curves across the arch faces are so slight that chords, as determined in Fig. 221, are drawn to represent the joint lines.

PLATE VI.

Face page 216.

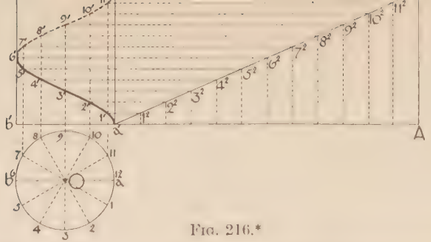
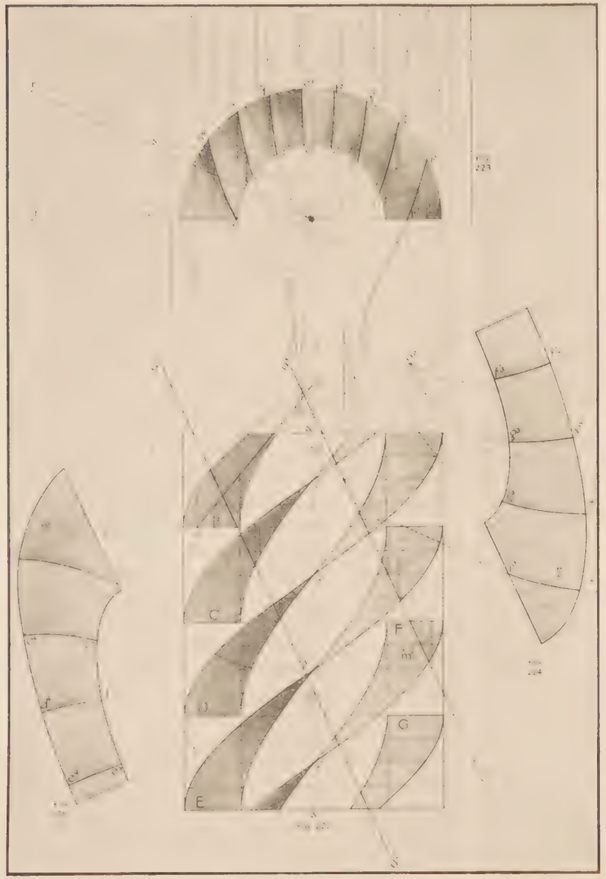


Fig. 216.*



Figs. 222, 223, 224, and 225.*

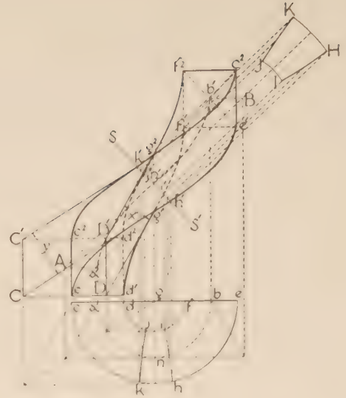
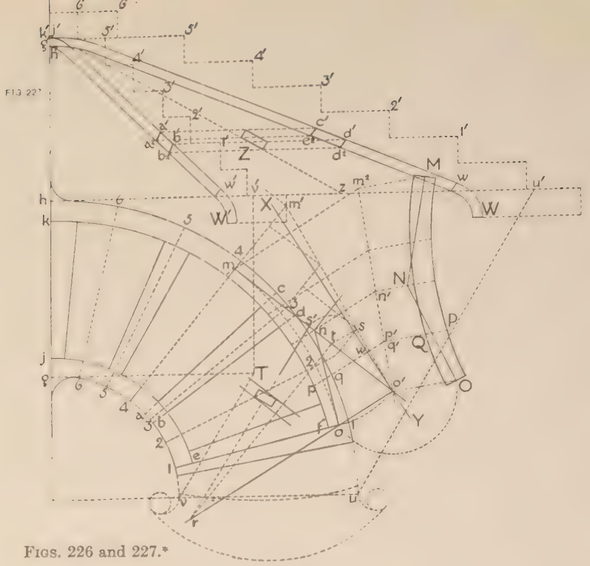
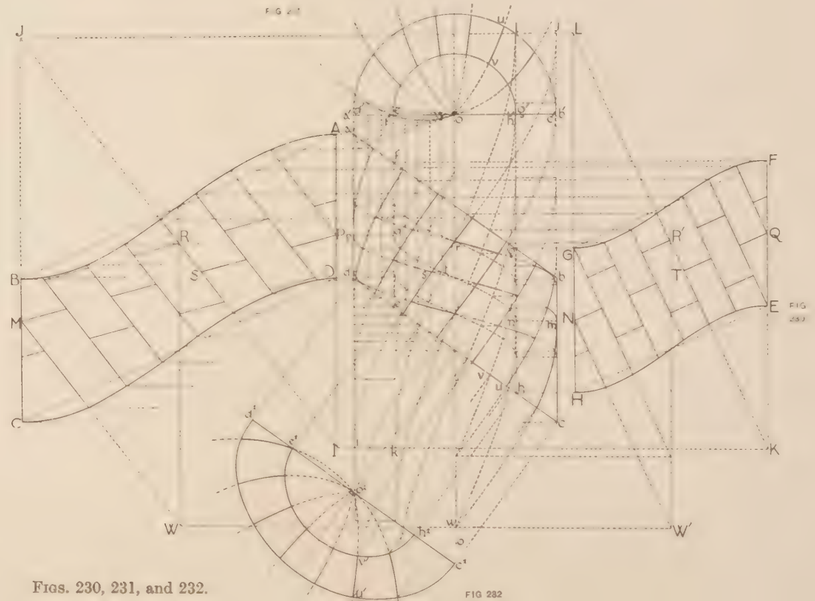


Fig. 219.*



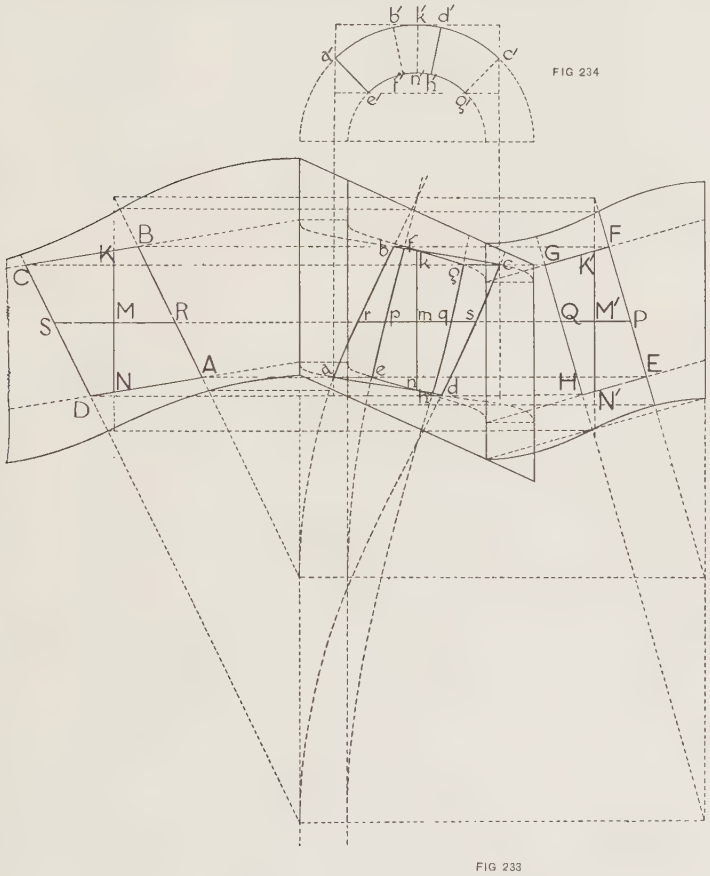
Figs. 226 and 227.*



Figs. 230, 231, and 232.

Geometry of the Skew Arch Voussoir.

Figs. 233 *and* 234.—In Fig. 233 is shown the inverted plan of a voussoir, determined precisely as in the last example. The voussoir is arranged with equal parts on each side of the vertical plane containing the axis of the cylindrical arch, and is made extremely large, compared



FIGS. 233 and 234.*

to the arch radius, to emphasise the nature of the bounding surfaces. The helical arrises are extended to assist in comparing with Fig. 230. The straight lines *ae*, *bf*, *gc*, and *dh* are the plans of the lines of intersection between the coursing and heading spirals. These lines are of equal length, being normals through the arch thickness. The helical curves *ab*, *cd*, *bc*, and *ad* are in the extrados of the arch and bound the

outer cylindrical surface of the voussoir. The helical curves *ef*, *gh*, *fg*, and *ch* are in the intrados and bound the inner cylindrical surface. The surfaces *abfe*, *edgh*, *bcfg*, and *adeh* are all helicoidal and each is bounded by two helices, one on the intrados and one on the extrados, and two straight line normals, each of the latter having one extremity in the intrados, the other in the extrados. All lines in the cylindrical surfaces of the voussoir parallel to the axis, as *kn*, are straight lines, and all lines across the helicoidal surfaces perpendicular to the axis, as *ae*, *bf*, *gc*, and *hd*, are straight lines. Let the straight lines *bf* and *dh* be produced to meet perpendiculars through *a* and *c*, as shown; then the rectangle so formed may be regarded as the plan of the material from which the voussoir may be worked. A projection of the voussoir on a plane perpendicular to the axis of the arch cylinder is shown in Fig. 234, and the circumscribing rectangle through c^1, g^1, a^1, and c^1 may be regarded as the elevation of the material out of which the voussoir is to be taken.

A Method of Working the Voussoir.

Same Figures.—In Fig. 233 *EFGH* is the development of the inner cylindrical face of the voussoir, *ABCD* a development of the outer cylindrical face, *KN* and K^1N^1 being the developments of the straight lines, one on each face, shown in plan at *kn*.

Prepare flexible moulds to the developments *EFGH* and *ABCD*, and on each mark the development of *KN*. Prepare also a mould to $a^1d^1h^1e^1$, Fig. 234, and on it mark the straight line k^1n^1.

True up the end surfaces of the material from which the voussoir is to be worked, making them perpendicular to the straight line *KN*; the length does not require to be exact. From the extremities of the line *KN* scribe straight lines across the end surfaces perpendicular to the face containing *KN*. Apply the moulds shaped to $a^1d^1h^1e^1$ to each of the end surfaces, placing the line k^1n^1 on the scribed line, reversing the mould as shown in Fig. 234 from $a^1d^1h^1e^1$ on one end to $b^1c^1f^1g^1$ on the other end. When the end moulds have been scribed on the end surfaces, the cylindrical faces of the voussoir may be worked, any required number of straight drafts being worked parallel to *KN*. The flexible moulds *ABCD* and *EFGH* may now be scribed on the cylindrical faces, the lines *KN* and K^1N^1 giving the positions relative to each other and to the end surfaces.

Since the intersections of the joint surfaces of the voussoir are straight, they may be worked first, and other straight drafts, as many as deemed necessary, may be determined by drawing parallels and perpendiculars to *KN*; thus *Q* and *S*, one on each mould, *N* and N^1,

one on each mould, give the directions of straight drafts across the winding surfaces.

There are several other methods, requiring elliptic moulds, of working the voussoir, but the student should now be able to evolve his own method.

It is interesting to note that the rail of the helicoidal framing, Figs. 226 and 228, is theoretically a solid similar in all respects to the skew arch voussoir, the ends of the finished rail corresponding to the cylindrical surfaces, the faces and edges of the rail corresponding to the helicoidal joint surfaces of the voussoir. The curved style, Fig. 229, is a similar solid, but the ends are made plane surfaces for purely practical reasons; theoretically they too should be helicoidal to make the joint lines normal to the arris helices.

Projections and Developments of a Skew Arch.

Figs. 235 *to* 240.—Fig. 235 is the plan of a skew arch with a number of voussoirs left out to show the joints on the intrados, some of the twisted joint surfaces, and one of the toothed abutments.

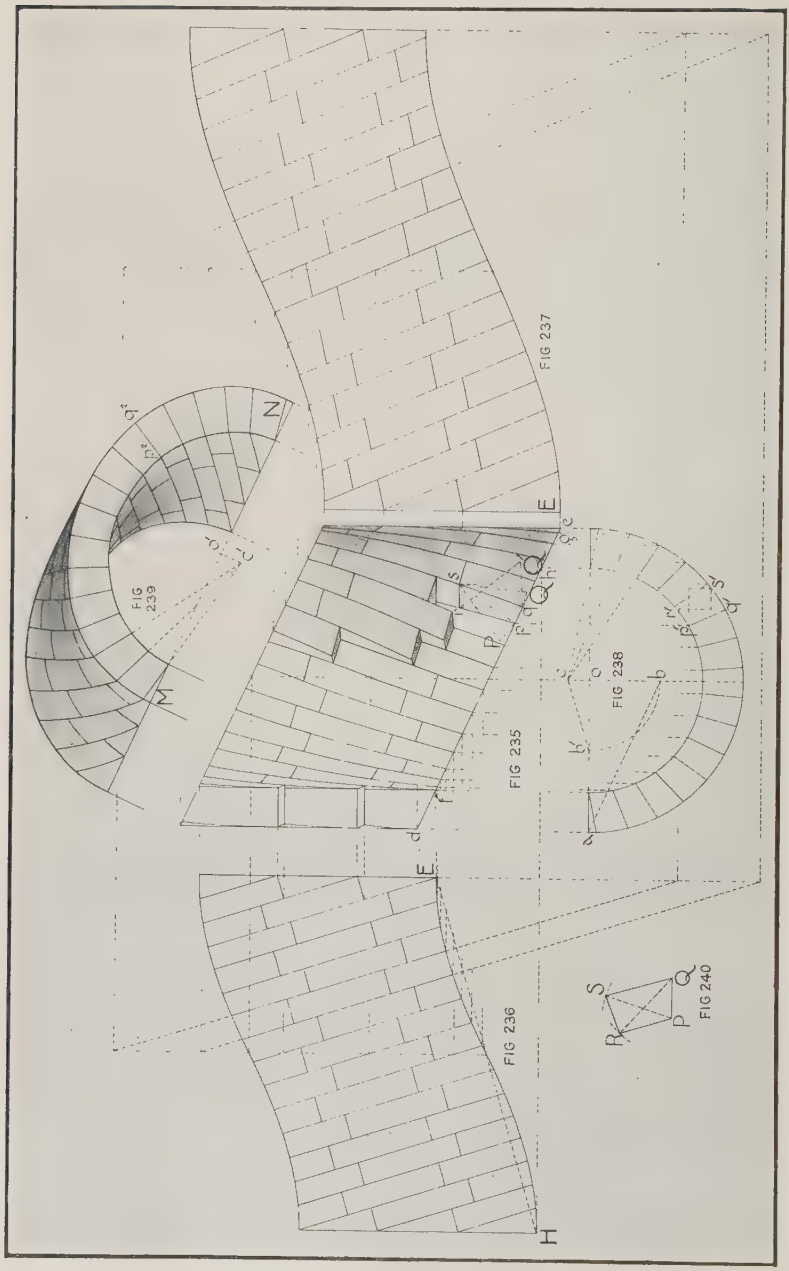
The development of the intrados is given in Fig. 236, and of the extrados in Fig. 237. The joint lines in plan and development are determined as in Fig. 230, the method of bonding the voussoirs being a purely practical consideration. The triangles about the straight line edges of the developments are formed on the upper courses of the abutments, but are actually in the arch surfaces.

Fig. 238 is an elevation of the arch on a plane perpendicular to the axis, in which the joint line chords radiate to the point c, this point being determined by drawing ab parallel to de, turning b about o to b^1, and drawing b^1c perpendicular to the coursing joint lines of the developed intrados.

Fig. 239 is an elevation of the arch on a plane parallel to the arch faces, in which the chords of the joint lines radiate to c^1, o^1c^1 being equal to oc, Fig. 238.

The small sectors at the springing of the arch faces are formed with the toothing on the abutments, and no joint line would necessarily appear in the springing plane.

There are no geometrical problems involved in this example additional to those discussed in connection with preceding figures, the curves in Fig. 239 being projections of the joint line helices. Fig. 240 is an approximate development of the joint surface $pqrs$ (Fig. 235) and is obtained by finding the true lengths of the straight lines joining r to q, p to s, and q to s, the elevations of the points being p^1, q^1, r^1, and s^1.



FIGS. 235, 236, 237, 238, 239, and 240.*

pP is perpendicular to ps and is equal to the difference in the heights of the points p and s ; qQ is perpendicular to qs and equal to the difference in the heights of q and s ; qQ^1 is perpendicular to qr and equal to the difference between the heights of q and r, the required difference being taken from Fig. 238. The quadrilateral, Fig. 240, may now be plotted, PQ being straight and equal to p^2q^2 (Fig. 239), and RS is straight and equal to the mean thickness of the arch r^1s^1 (Fig. 238). The joint surfaces are, as already stated, curved and undevelopable, therefore $PQRS$ is only approximate ; but for the purpose of working the surfaces of the short voussoirs to form the plane faces of the arch, moulds determined in this way would be sufficiently accurate.

Cylindrical Vaulting.

Figs. 241 *to* 248.—Fig. 241 is the plan and sectional elevation of the intersection of two semi-cylindrical headed vaults of equal radii. On the left of the centre line the plan is inverted, and the supposed material is stone. On the right of the centre line the supposed material is wood. The diagram is arranged in this way simply to conserve space ; it must be understood that in practice the two materials would not be combined in the same vault.

The lines of intersection between the branches of the vault are called groins, and in stone vaulting only the groin voussoirs are of geometrical interest. Consider the voussoir $ab \ldots j$ in plan, $a^1b^1 \ldots j^1$ in elevation, Fig. 241, in conjunction with the enlarged pictorial view, Fig. 243. The surfaces GDJ and CDJ of the voussoir lie in the outer cylindrical faces of the vault, DJ being part of the line of intersection between the branches of the vault.

Similarly the surfaces $IBFH$ and $ABFE$ are in the inner cylindrical faces of the vault, BF being part of the line of intersection. The surfaces $IGJH$ and $AEJC$ are in the vertical heading joint planes. The four remaining surfaces $GBDI$, $CBDA$, JEF, and JHF are in the radial coursing joint planes.

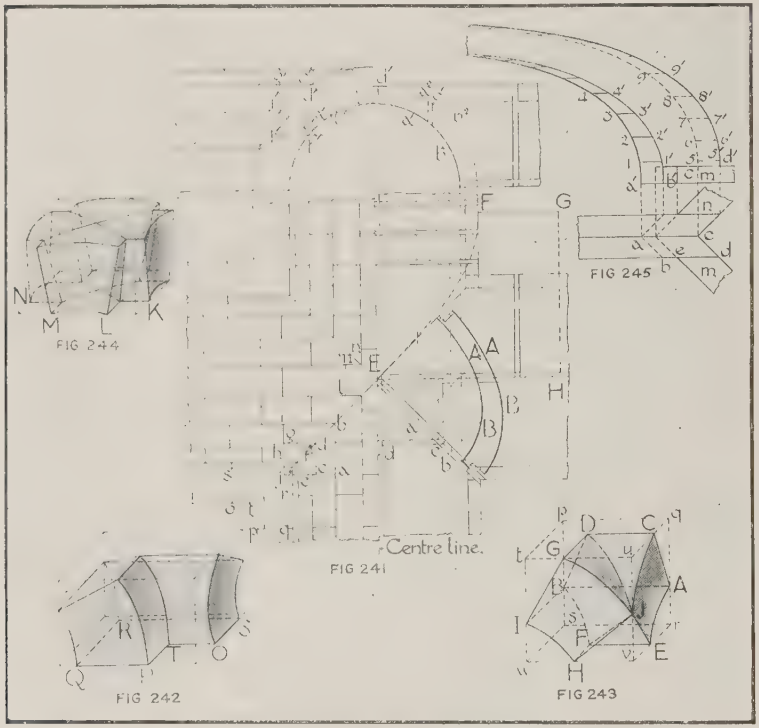
The dotted lines, *pqrs*, *tuvw*, Fig. 243, represent the arrises of the stone block from which the voussoir may be taken, the true shape and position of the vertical joint mould to apply to the block being shown at $a^1d^1e^1i^1$, Fig. 241.

Pictorial views of the groin springer and groin key stone are given in Figs. 242 and 244 respectively, the capital letters in these figures corresponding to the small letters in the inverted plan.

In considering the right-hand half of the figure it will be seen that the groin rib mould is half the elliptic section of the hollow semi-cylinder

which forms the head of one branch of the vault. All the ribs are formed with a lip on each end, the lower lips of the ribs fitting against the inner edges of the wall plates, the upper lips passing half way across the under edges of the ridges; thus the four groin ribs would form a mitred intersection at *E*.

Since the groin rib must have half its thickness in each branch of the vault which intersects in its vertical centre plane, the rib should be



Figs. 241, 242, 243, 244, and 245.

made in two thicknesses, each half being bevelled along its upper and lower edges to conform to the cylindrical surfaces. In Fig. 245 part of the finished groin rib is shown in plan and elevation. The elliptical curves on the lower edge are identical, and the elliptical curves on the upper edge are identical. Therefore the horizontal straight lines a^1b^1, $1\,1^1$, $2\,2^1$ . . . $4\,4^1$, etc., across the lower edge of half the thickness, and c^1d^1, $5\,5^1$, $6\,6^1$ . . . $9\,9^1$, across the upper edge of half the thickness, are all equal. Since the centre plane and the two face planes are each identical with the shape of the mould, the bevel, or backing, is set out

by marking the mould on one face of the material forming half the thickness of the rib, then moving the mould horizontally through the distance a^1b^1. The outline curves a^1, 1, 2 . . . 4 and d^1, 6^1, 7^1 . . . 9^1 are thus obtained, and after the material has been cut to these curves, the mould may be marked on the other side of the material immediately opposite its second position on the first side.

To obtain the mitre lines on the horizontal lagging forming the cylindrical surfaces, the latter may be developed as shown by the dotted lines *EF*, *FG*, and *GH*, Fig. 241.

The lengths and bevels for the short ribs which spring from the groin rib, as *dc*, right of Fig. 241, may be determined by projecting on to the elevation rib ; thus d^1c^1 is the true length of the rib, *dc* in plan, and the upright line from c^1 across the face of the rib is the cutting line. The edge cut may be found by setting the thickness of the rib horizontally back from c^1, as shown by the dotted line to the left of c^1.

The preceding example of cylindrical vaulting is the most simple that may be taken, but nevertheless it embodies all the geometrical principles involved in this type of construction. The slight modifications required when the branches of the vault vary in width are shown in Figs. 246, 247, and 248. On the left of centre line *T* stone construction is represented, on the right, timber construction. The branch of the vault having the centre line *S* is semicircular, the other branch, centre line *T*, being semi-elliptical.

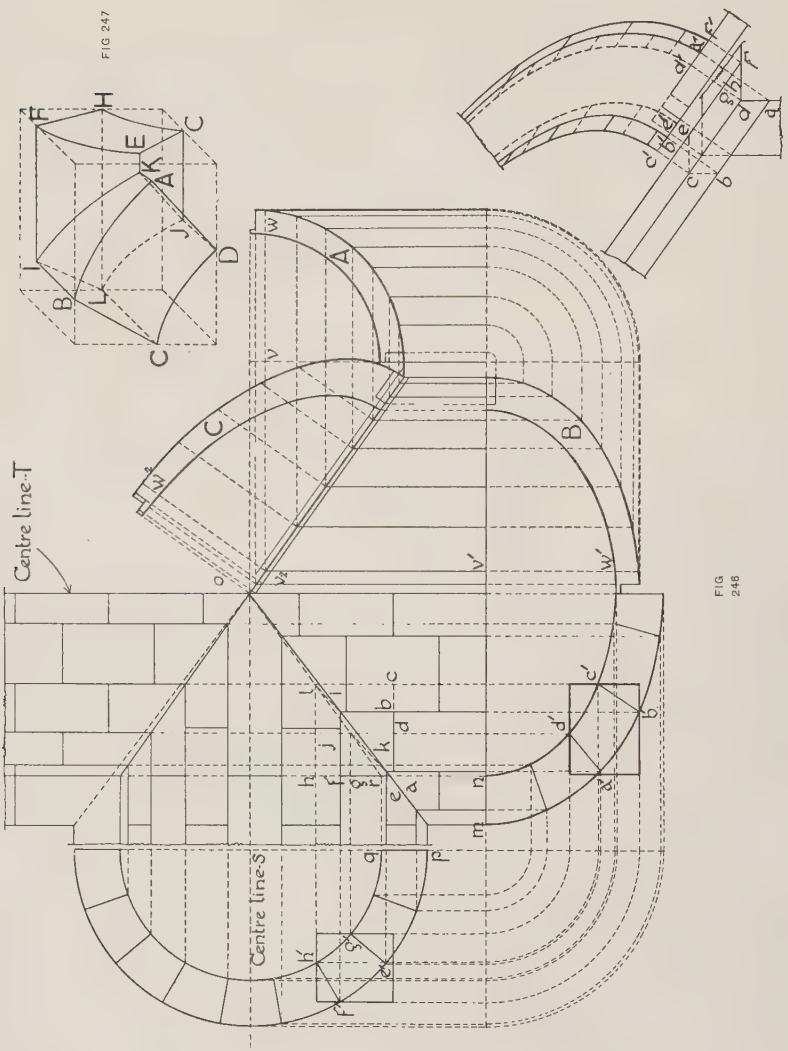
Above the centre line *S*, the joint lines on the intrados of the stone vault are shown, below *S* the extrados joint lines are drawn firm.

To keep the supporting walls of the two branches equal in thickness, the base of the springer *pq* in the semicircular vault must equal the base of the springer *mn* in the elliptic vault, the result being that the inner and outer arrises of the groin do not lie in the same vertical plane, *ro* being the plan of the inner arris and *ko* the plan of the outer arris. Also, in keeping the coursing joint lines of the two branches in the same horizontal planes, the joint planes in the elliptic head are not normal to the elliptical surfaces. The plan and vertical joint sections of one of the groin voussoirs are lettered to correspond with the enlarged pictorial view of the voussoir, Fig. 247. The size of the material containing the voussoir and the positions of the moulds are shown in the two figures.

On the right of the centre line *T*, Fig. 246, the construction lines should provide sufficient explanation of the method of obtaining the rib moulds.

NOTE.—$vw = v^1w^1 = v^2w^2$, and similarly for the other ordinates.

The plan and elevation of the lower end of the groin rib are shown in Fig. 248, and it will be seen that, because the centre plane of the rib does not bisect the angle between the wall plates, the two half



Figs. 246, 247, and 248.

thicknesses of the rib are unequally backed. Again, however, the centre plane and the face planes are each equal to the mould in every respect, and the bevels are obtained by sliding the mould horizontally through the distance *gh* for the forward half, and *gf* for the other half.

CHAPTER XIV

THE CONE

Some important properties of the cone have already been introduced in the previous chapters and it is only intended here to consider one or two typical examples of practical problems involving this solid.

The Projections and Development of Lines in the Surface of a Right Circular Cone.

Fig. 249.—The plan and elevation of a cone, vertex V, are shown in the figure. Let *abcd* be the plan of a line lying in the surface of the cone ; then if $v1$, $v2$, etc., are the plans of generators intersecting the plan of the line, the elevation of the line may be obtained by drawing the elevations of the generators and projecting the points of intersection. Thus $v5$ is the plan of a generator whose elevation is $v^1 5^1$; then if a, the plan of the point of intersection between $v5$ and the plan of the line, be projected to a^1, on $v^1 5^1$, a^1 is a point in the elevation of the line. Any number of points may be found in a similar manner.

To develop the surface of the cone containing the line. With v^1 as centre, the true length of a generator as radius, draw the circular arc 0^1, 1^2, 2^2 . . . 0^2. Draw the plans and elevations of any number of generators, preferably, not necessarily, to meet the base circle in equidistant points, as at 1, 2, 3 . . . 11. Set out along the arc $0^1 0^2$ a series of distances equal in length and number to the distances into which the base circle is divided by the generators, and join the points of division to v^1. Then the area lying between the straight lines $v^1 0^1$, $v^1 0^2$, and the arc $0^1 0^2$ is the developed surface of the cone and the straight lines $v^1 1^2$, $v^1 2^2$ are the developed generators. Through a^1, b^1, c^1, and d^1, the points in which the elevations of the generators cut the elevation of the line, draw straight lines parallel to the base of the cone to cut the outline generators in m^1, n^1, o^1, and p^1. With v^1 as centre, and $v^1 p^1$, $v^1 o^1$, $v^1 n^1$, and $v^1 m^1$ as radii, draw circular arcs across the development. Then, if the horizontals from m^1, n^1, o^1, p^1 be regarded

as the elevations of circular sections of the cone, the arcs across the development are the developments of these circles; and since a^1, b^1, c^1, and d^1 are the elevations of points of intersection between the generators, the circular sections, and the line, the corresponding

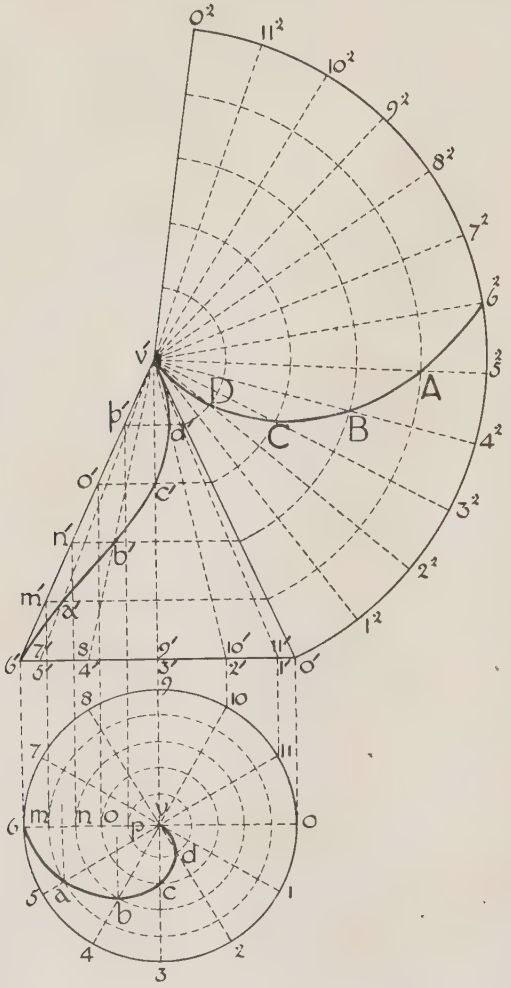
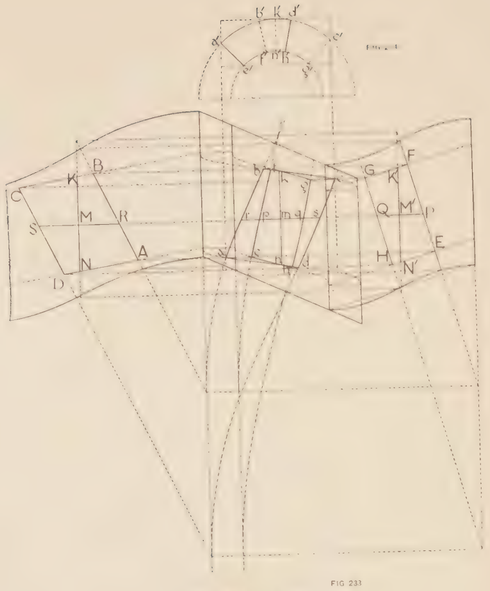


FIG. 249.*

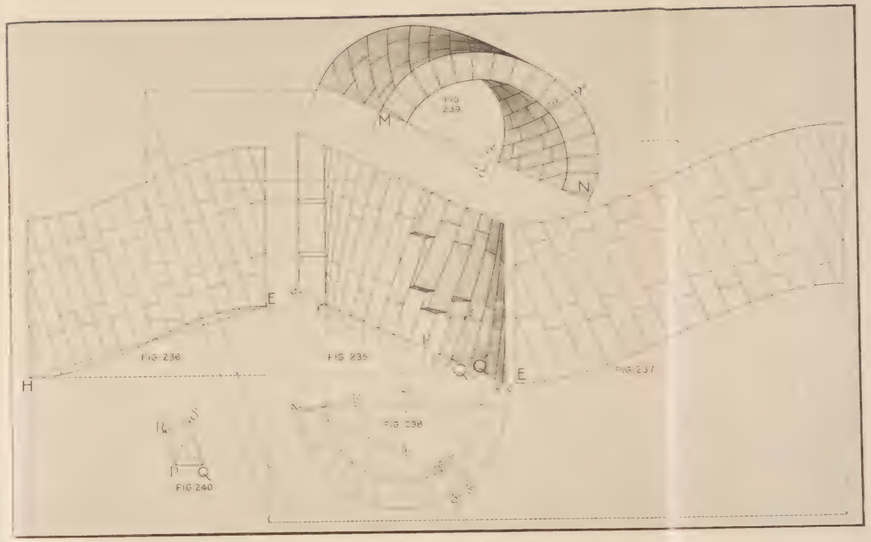
intersections may be found in the development and the developed line may be drawn.

The circles shown in plan through m, n, o, and p are the plans of the horizontal sections shown in elevation and development, and it

PLATE VII.



Figs. 233 and 234.*



Figs. 235, 236, 237, 238, 239, and 240.*

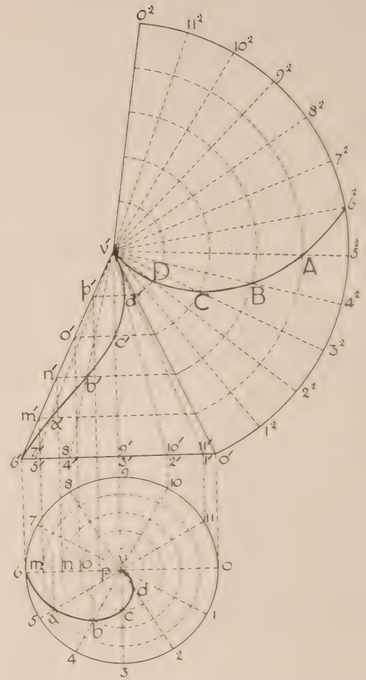


Fig. 249.*

Next page 228.

should be noted that these circles provide a second method of determining the projections of points on the surface of the cone.

The plan and elevation of a cone may be determined from its development by the following formula: $R : r = 360° : a$; where R equals the generator length, r equals the radius of the base, and a equals the plane angle of the development. Thus $v^1o^1 : vo = 360°$: angle $o^1v^1o^2$.

To determine the shortest distance between two points on the surface of a cone, measured along that surface; develop the portion of the surface lying between the two generators through the points, determine the positions of the points on the developed generators, draw a straight line between these positions, and bring the line into the surface of the cone by reversing the process explained in connection with Fig. 249.

Section of a Cone by an Oblique Plane.

Fig. 250.—The plan and elevation of a right circular cone, vertex V, and the traces, vth, of an oblique cutting plane are given in the figure. The projections of the elliptic section 1, 2, 3 . . . 9 and 1^1, 2^1, 3^1 . . . 9^1 are determined as follows: Draw ab through v and perpendicular to ht. Project a to a^1 and b to b^1, and join a^1b^1; then a^1b^1 is the elevation of the line of intersection between the given plane vth and a vertical plane on ab. Draw the elevations of the two generators of the cone which lie in this vertical plane to intersect a^1b^1 in points 1^1 and 5^1, and project these points to 1 and 5 in ab; then $1\,1^1$ and $5\,5^1$ are the projections of two points in the required section and 1 5 is the major axis of the plan ellipse. Through c, the mid point of 1 5, draw ce parallel to ht. Project e to e^1 and draw e^1q^1 parallel to xy. Draw vs parallel to XY and project q^1 to intersect it in q. With v as centre, vq as radius, cut ec in 3 and ec produced in 8, and project 3 and 8 to 3^1 and 8^1 in e^1q^1; then $3\,3^1$ and $8\,8^1$ are projections of points in the section line, and 3 8 is the minor axis of the plan ellipse. Draw f^1p^1 and d^1r^1 parallel to XY, project p^1 and r^1 to p and r respectively, and project d^1 and f^1 to meet XY in d and f. Draw straight lines from d and f parallel to ht, and with centre v, radii vp and vr, cut the parallels from f and d in points 4, 7 and 0, 9 as shown. Project 0 and 9 to 0^1 and 9^1 in d^1r^1, and 4 and 7 to 4^1 and 7^1 in f^1p^1; then $4\,4^1$, $7\,7^1$, $0\,0^1$, and $9\,9^1$ are projections of points in the section line, and any number of points may be determined in a similar manner.

For, consider the points 4 and 7. Since $f7$, f^17^1 are projections of a horizontal line in the plane vth, and the circle, centre v, through P is the plan of the section of the cone by the horizontal plane containing

this line, the points of intersection between the line and circle are in the plane and in the surface of the cone, and are therefore in the line of intersection between the plane and the cone. To determine the points in which the elevation of the section touches the outline generators, produce *sv* to meet *ht* in *g* and project *g* to g^1 in *XY*. Then if *gs* is the plan of a line lying in the plane, its elevation is parallel to *vt*, *gs* being parallel to *XY*; and since *GS* is in the plane containing the

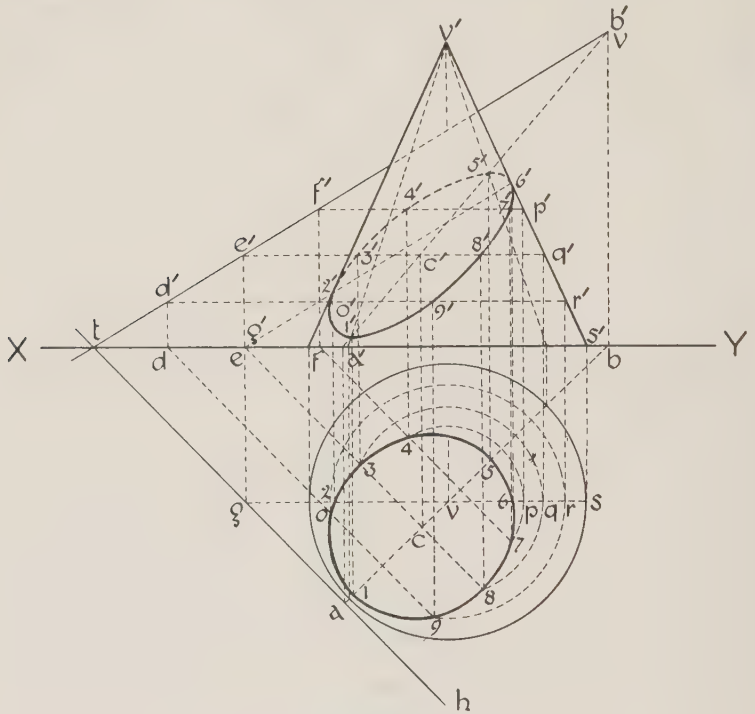


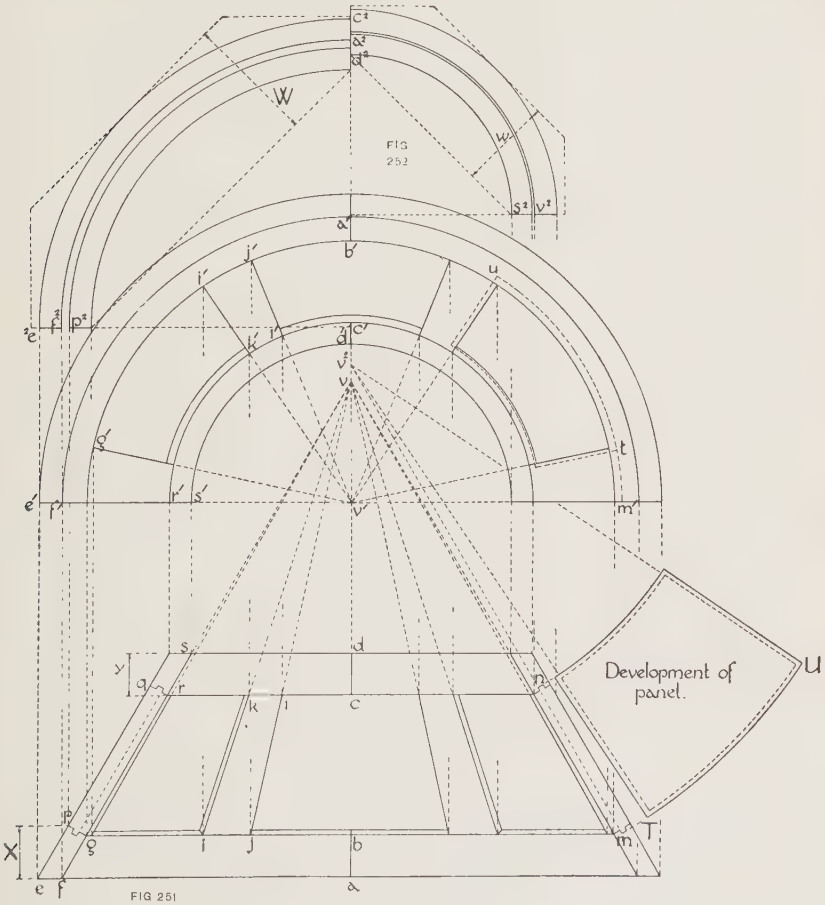
Fig. 250.*

outline generators, the points 2^1 and 6^1 in which the elevation of the line intersects the outline generators are in the elevation of the elliptic section.

Conical Framing.

Figs. 251, 252, *and* 253.—Fig. 251 is the elevation and inverted plan of the circular head of a splayed soffit lining. The inner faces of the rails and curved styles lie in the surface of a cone, the projectors of whose vertex are *v* and v^1. The faces of the panels are in a second conical surface, parallel to the first, the plan of whose vertex is v^2.

In making the projections of the lining the plan outline is first drawn. The width *fg* of the inner faces of the styles, the thickness *qr* of the framing, and the position of the panel are set out to conform to the straight framing of the jamb linings upon which the soffit lining is assumed to rest. The elevation may now be drawn, the inner arrises



FIGS. 251 and 252

of the rails being made to radiate to v^1. The plan can now be completed.

The panels are formed by glueing staves on to the back of a veneer, a conical drum or mould being made to the shape of the face of finished panel and the process of staving being carried out on the drum. The shape of the veneer is obtained as follows: Set out the depths of the

panel grooves as shown at m and n, and continue the line mn to meet the centre line in v^2. With v^2 as centre, radii v^2n and v^2m, draw circular arcs, and on the arc from m set out the distance $T\,U$ equal to the arc tu in elevation, the latter being determined by projecting m to m^1 and turning m^1 about v^1. The dotted radials from t and u represent the depths of the grooves in the rails. The dotted lines on the panel development are the sight lines, two of which radiate to v^2, the outside lines being drawn parallel.

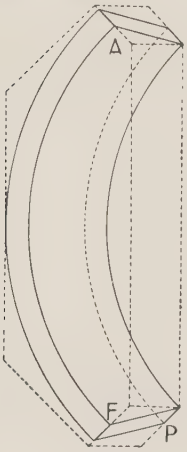


Fig. 253.

Each of the curved styles is made in two sections, and the required moulds for cutting them out of the plank may all be obtained from the elevation. The thickness of the material required for the front style is marked X in plan, and the required width is marked W in Fig. 252.

In the latter figure the positions of the moulds, one on each side of the plank, are shown, the moulds being shaped to the elevation curves; the inner edges of the styles are parallel to the outer edges when first cut out of the plank. It will be seen that all the moulds have a width equal to ef, Fig. 251. The letters in the two figures show corresponding points.

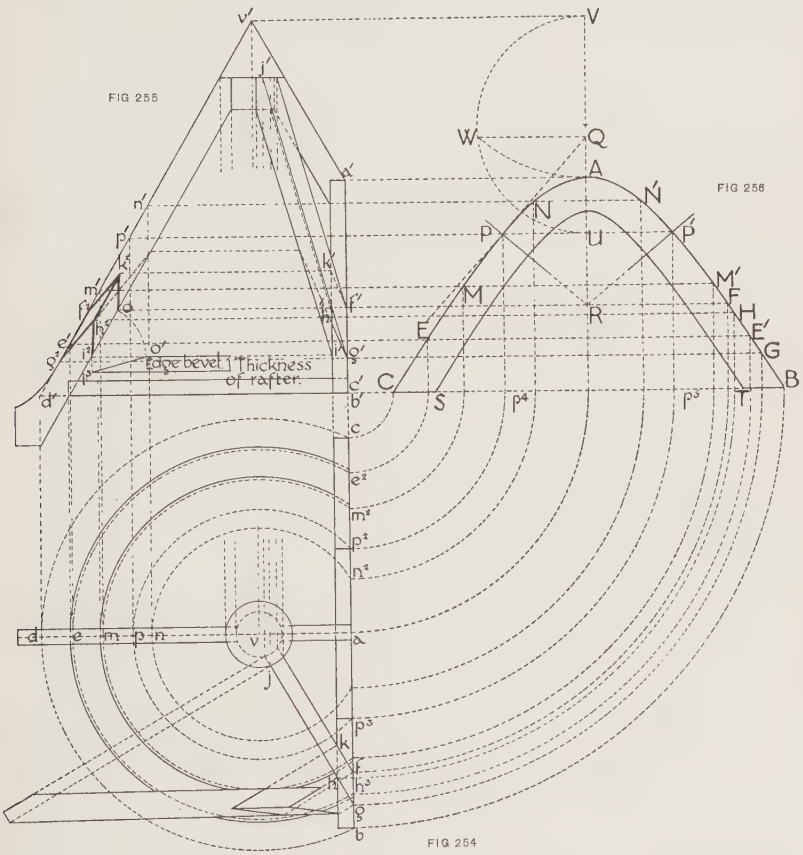
A pictorial view of one section of the style as it lies in the plank is shown in Fig. 253.

Intersection of a Conical Roof with a Vertical Wall Surface.

Figs. 254, 255, *and* 256.—Figs. 254 and 255 are the projections of the wall plate, a full length rafter, two short rafters, the wall curb, and an apex boss in a conical roof. The roof intersects a vertical wall surface along the line abc, $a^1b^1c^1$. The edge of the curb is at right angles to the face, and the short rafters birdsmouth on to it. The problems involved are: to determine the hyperbolic outline of the curb, the normal joint lines, and the bevels for birdsmouthing the rafters. The required hyperbolic outline is ACB, Fig. 256, and is determined as follows: Draw ad perpendicular to bc (254) and draw any number of circular arcs dc, ee^2, mm^2, pp^2, nn^2 to intersect bc. Project d, e, m, p, n to d^1, e^1, m^1, p^1, n^1 (255), and taking pp^2 as an example draw a horizontal from p^1, turn p^2 and p^3 about c^1 to meet d^1c^1 produced in p^4 and p^5, and draw perpendiculars from p^4 and p^5 to meet the hori-

zontal from p^1 in P and P^1; then P and P^1 are in the required outline. Draw the curve ST parallel to the hyperbola to represent the required width of the curb.

To determine the normal joint lines at P and P^1. Project v^1 on to the axis of the hyperbola to obtain V and draw the ordinate PP^1



Figs. 254, 255, and 256.*

to intersect the axis in U. Draw a semicircle on VU, and with centre V, radius VA, cut it in W. Draw a straight line from W to intersect VA perpendicularly in Q; then Q is in the tangents to the curve at P, P^1, and the normals PR, P^1R are at right angles to these tangents.

This construction has been explained in the chapter on conics.

To determine the angles for cutting the feet of the rafters.

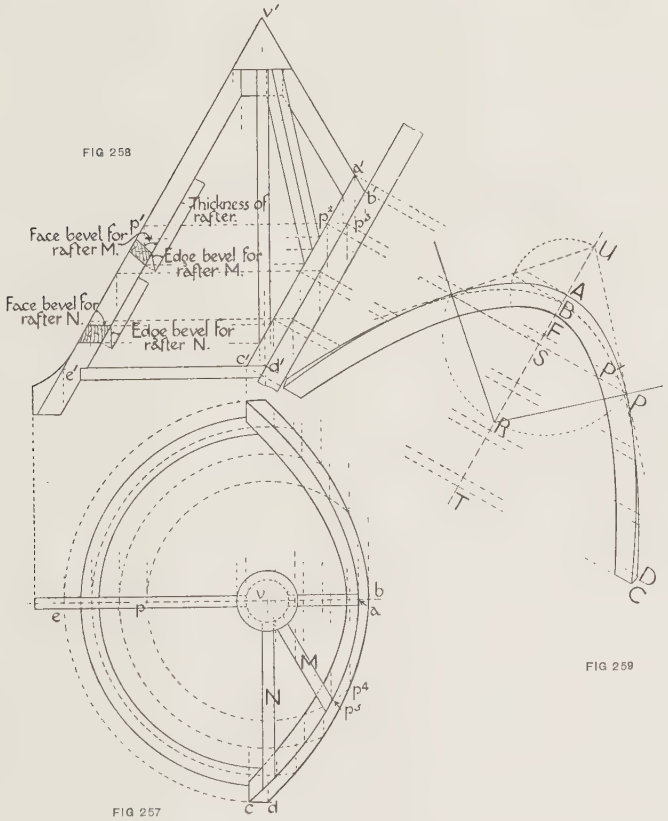
Draw the plan of the required rafter, as jg (254), to intersect the plan of the curb in points g, h, f, k. Project h perpendicularly through the curb to h^3, and by turning about c^1, as explained in connection with point P, convey g, h^3, f to G, H, F (256). The elevation of the rafter may now be obtained in Fig. 255 by projecting horizontally from F, H, G to $f^1h^1g^1$, the upper end of the rafter being projected from the plan. The face bevels may be determined by projecting the lower end of the rafter to g^2, i^2, h^2, f^2, and k^2, the lines i^2h^2 and ok^2 being perpendicular—since the face of the rafter and the face of the curb both lie in vertical planes. The true length of the rafter is also obtained by this method, since the process is equivalent to turning the rafter around in the roof surface until it is parallel to the plane of projection. The edge bevel is found by turning o about i^2 to o^1 and making the perpendicular i^2i^3 equal to the thickness of the rafter. A second method of obtaining the length and bevels for the rafter is shown in Fig. 254 and consists of making a projection on to a plane parallel to the face of the rafter, the heights of the points being taken from Fig. 255.

Intersection of a Conical Roof with a Plane Roof Surface of Equal Inclination.

Figs. 257, 258, *and* 259.—In this example it is arranged to have the edge of the curb of parabolic outline, in the conical surface containing the upper edges of the rafters ; therefore the edge of the curb is a conical surface bounded by two parabolic curves, the curves being parallel sections of the conical roof.

In Fig. 258, $a^1b^1c^1d^1$ is the edge elevation of the curb. The method of obtaining the plan, Fig. 257, and the true outline, Fig. 259, will be sufficiently explained by considering the points A, B, C, D, and P. Draw UT (259) parallel to a^1c^1 (258) at any convenient distance, and draw a^1A and b^1B perpendicular to a^1c^1 ; then A and B are the vertices of the two parabolic outlines, and AT is the common axis. From e^1, in the horizontal line c^1d^1 produced, drop a perpendicular to e, ev being drawn parallel to e^1d^1 ; and about v as centre turn e to cut perpendiculars from c^1 and d^1 in c and d. Draw c^1C and d^1D perpendicular to AT and make the distance of D and C from AT equal to the distance of c and d from ae ; a being in ev produced and in the perpendicular from a^1. b is also in ev produced and is the plan of B. Now let $p^1p^2p^3$ be in a horizontal section of the conical roof. Then if p^1 be projected to p in ev, the circle through p, centre v, is the plan of the section, and perpendiculars from p^2 and p^3 cut the circle in p^5

and p^4, points in the plans of the arrises of the curb. Draw p^2P and p^3P^1 perpendicular to a^1c^1, and make the distances of P and P^1 from AT equal to the distances of p^5 and p^4 from ea respectively; then APC are points in the upper arris and BP^1D points in the lower arris. Other points in the plan and true shape of the curve are determined



Figs. 257, 258, and 259.

in a similar manner. The lengths and bevels of the rafters are determined as in the previous example.

To determine the joint normal at P, make AU equal AS and draw PR at right angles to PU; then PR is the required normal.

If other normals are required, the focus F may be found and used by making AF equal half SR; then for other normals make FR equal FP, as explained in connection with the parabola.

It will be seen that the edge of the curb is not square to the face, excepting at the points of intersection of the two parabolas which occur just below the normals, and since the two parabolas are unequal, two moulds, one for each side of the material, would be required for shaping the edge. The inner edge of the curb is drawn at will and may be square to the faces.

CHAPTER XV

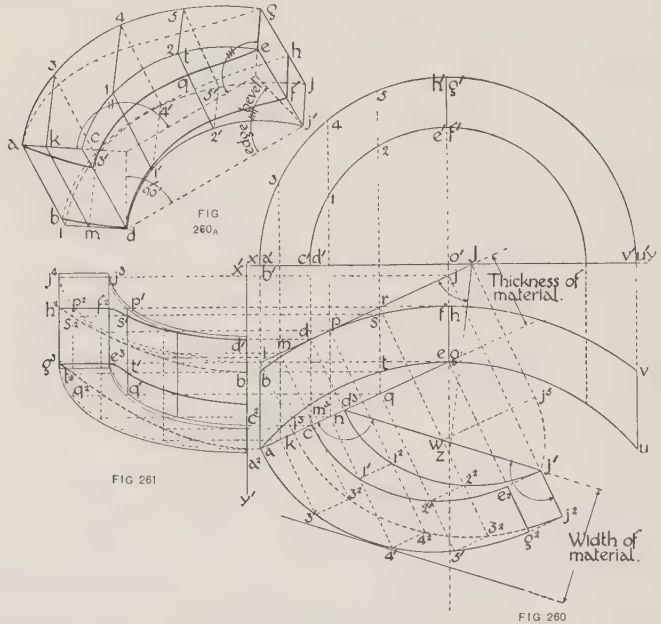
THE GEOMETRY OF DOUBLE CIRCULAR WORK

ARCHES, window and door frames, etc., which are circular in elevation and stand over circular plans, are referred to as circle-on-circle work. Arches of this type present no geometrical intricacies, and frames small enough to be made in two sections are also comparatively simple. There are, however, some interesting problems to be solved in connection with large frames comprising three or more sections, and a careful study of the more simple examples will be of assistance when considering the more difficult cases.

Example 1. *Figs.* 260, 260*a*, *and* 261.—The two circular segments *au* and *bv*, centre *o*, Fig. 260, are the plans of the vertical cylindrical faces of the head of a solid frame whose semicircular elevation is shown above *XY*. The frame has parallel jambs and the head is in two sections, *EFGH* being the joint plane.

The head is a solid formed by the interpenetration of two hollow right cylinders, the plan cylinder having its axis *o* vertical, the axis o^1 of the elevation cylinder being horizontal. The thickness of the material required to contain one section of the head is determined by drawing the chord *ag*, and the tangent *ij* parallel to the chord; the distance between the chord and tangent is the required thickness. The moulds required to obtain the elevation curves are the plane sections of the horizontal cylinder by the vertical face planes of the material, that is, by the vertical planes on *ag* and *ij*; and since these planes are parallel, the two moulds will be equal. The elliptic outlines of the moulds may be determined by drawing the horizontal ordinates *km*, *np*, and *qr* parallel to oo^1 through the thickness of the material and producing them to cut the elevation curves in points 1, 2, 3, 4, and 5, and drawing $k3^1$, $n1^1$, $n4^1$, $q2^1$, $q5^1$, ee^2 and eg^2 perpendicular to *ag* and equal to the heights of the corresponding points above *XY*. Then curves through *a*, 3^1, 4^1, 5^1, g^2 and *c*, 1^1, 2^1, e^2 are the elliptic outlines of the moulds. Project *i*, *m*, *p*, *r*, and *j* perpendicularly to *ag*, to i^3, 1^2, 2^2, j^1, 3^2, 4^2, 5^2, and j^2 to obtain the relative positions of

the two moulds when applied one on each side of the plank; $3^1 3^2$, $1^1 1^2$, etc., are all parallel to ag. Join $d^3 j^1$ and parallel to it draw the tangent to ag^2; then the distance between the chord $d^3 j^1$ and the parallel tangent is the width of the material required. When the material is being cut square out of the plank, the cuts must be made along the curves ag^2 and $d^3 j^1$, the moulds being tacked together in the positions shown for marking the cutting lines. The lines $1^1 4^1$, $2^1 5^1$, etc., should be kept on the moulds when working the horizontal



Figs. 260, 260*a*, and 261.*

cylindrical surfaces to give the straight line directions for applying the straight edge.

To work the plan curves, or vertical cylindrical surfaces, the lines $1^1 1^2$, $2^1 2^2$, $3^1 3^2$, etc. require to be drawn across the worked surfaces before the moulds are removed, so that the distances qt and sr may be set out from the plane faces along the lines $2^1 2^2$, $5^1 5^2$, the distance jf along the lines $e^2 j^1$, $g^2 j^2$, and similarly for the other positions, including km^3. Smooth curves around the cylindrical surfaces through the points thus obtained are the arrises to be formed by working the vertical cylindrical faces, the points also giving the directions for applying the straight edge when these faces are being formed.

Fig. 261 is an elevation of the section lying in the plank

after the material has been removed to the edges of the face moulds. The straight line directions are indicated across the four surfaces.

Although the straight line ends of the moulds are in the joint planes, they are not stable enough to give the required accuracy for working the joints; the latter are therefore worked before the moulds are applied, the three necessary angles for setting the bevel being ad^3j^1, $d^3j^1j^2$, and iJz; the last mentioned being the upper edge bevel, the other two, face bevels. The two face bevels are determined when d^3j^1 is drawn, the edge bevel being as follows: Produce or draw any line square through the thickness of the material, as $r2^2$ to meet jg produced in z, and to cut d^3j^1 in w. Draw wj^5 parallel to ag and equal to wj^1. From j^5 draw a perpendicular to ag to meet ij produced in J. Join Jz. The positions for applying the bevels are shown in the sketch, Fig. 260*a*: by comparison with Fig. 260 the corresponding angles in the two figures may easily be found. In the sketch, $aceg$ is the position of the mould ace^2g^2 on the upper face of the plank, and $idfh$ is the position of the mould $cd^3j^1j^2$ on the under surface.

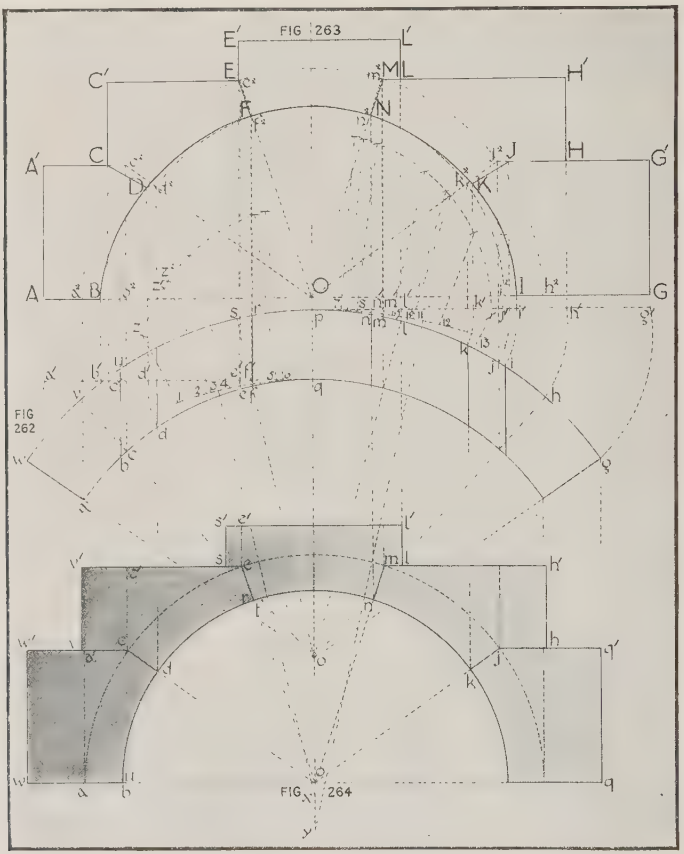
Circle-on-Circle Masonry Arch with Parallel Jambs.

Figs. 262, 263, *and* 264.—Fig. 262 is the inverted plan of a stone arch. The arch is segmental in plan and semicircular in elevation. The soffit is part of the surface of a horizontal cylinder; the faces are parts of the concave and convex surfaces of a vertical hollow cylinder. The soffit arrises are the lines of intersection between the cylindrical surfaces.

The elevation of the arch is shown in Fig. 264, and it will be seen that the extrados is stepped, the vertical plane faces of the stepping being normal to the surfaces of the vertical cylinder. The radial joints between the voussoirs are slightly curved surfaces, being bounded by two horizontal straight lines and two helical curves. The latter curves are too slight to be obtained in a small scale drawing, and are shown straight at cd, ef, mn, and jk in the elevation, Fig. 264. Further reference will be made to the radial joint surfaces after the method of producing them has been discussed.

Two face moulds and a bed mould are required in working each voussoir, the face moulds being the developments of the two cylindrical faces of the voussoir, the bed mould being the plan outline of the voussoir. The face moulds, shown in Fig. 263, are determined as follows: Draw the tangents pg^1 and qa^1 to the plan segments, Fig. 262, and develop the points a, b, c, d, e, f in the arc aq to a^1, b^1, c^1, d^1 e^1, f^1 in the tangent a^1q, and the points g, h, i, j, k, l, m, n in the arc pg

to g^1, h^1, i^1, j^1, k^1, l^1, m^1, n^1 in the tangent pg^1. The arcs aq and pg are developed by the method explained in Chapter I., problems 1 and 2, but the following reiteration may prove helpful: Determine the points x and y, such that ox equals half oq and oy equals half op. With x as centre, radius xq, draw the arc q-1. With y as centre, radius yp, draw



Figs. 262, 263, and 264.

the arc p-13. Draw the normal bisectors of the arcs, qa, qb, qc, qd, qe, and qf, to cut the arc q-1 in points 1, 2, 3, 4, 5, and 6 respectively. With 1 as centre, turn a to a^1; with 2 as centre, turn b to b^1, and so on in the corresponding order. Similarly, on the other side of the centre line draw the bisectors of the arcs pg, ph, pi, pj, pk, pl, pm, and pn to cut the arc, centre y, in points 13, 12, 11, 10, 9, 8, and 7. With 13 as centre, turn g to g^1; with 12 as centre, turn h to h^1, and so on to 7 as centre,

turn n to n^1. It should be noted that all the arcs contained in qa, to be bisected, have q as a common extremity ; therefore all the bisectors may be obtained by drawing any sufficiently large arc, centre q, and with the same radius and the other extremities of the arcs as centres, cutting this arc in points in the bisectors, as shown at z, z^1, z^2.

Through any point O in op produced draw a perpendicular AG to op. Project a and b on to this line to a^2 and b^2, and with O as centre draw the semicircles a^2h^2 and b^2i^2, Fig. 263. Divide either of these semicircles into as many equal parts as there are required voussoirs in the arch, and draw the radials d^2c^2, f^2e^2, n^2m^2, k^2j^2. Or, when the elevation has already been drawn and the joint lines drawn across the plan, as in the present example, $d^2c^2f^2e^2$, etc., may be obtained by projection ; the complete elevation may, however, be dispensed with, the joint lines in plan being obtained and developed after the semicircles, Fig. 263, have been drawn and divided. From the points a^1, b^1, c^1 . . . n^1 in the lines qa^1 and pg^1 draw perpendiculars to meet horizontals from a^2, b^2, c^2 . . . n^2 in points A, B, C . . . N. Join A to B, C to D, E to F, G to I, J to K, and M to N. Draw the horizontals from C, E, M, J and the perpendiculars from A, C, E, G. Project l^1 to LL^1 and h^1 to HH^1. The face moulds for the concave faces of the voussoirs are on the left of the centre line ; for the convex faces, on the right of the centre line. One half only of each mould required for the keystone is shown. The bed mould for the springers is the part of the plan, Fig. 262, between aw and dt. The bed mould for the keystone is the plan area between the radials at s and l. The bed mould for the intermediate voussoirs is the plan area $cvfr$.

The material is first worked to the bed mould so as to obtain the upright cylindrical surfaces around which the face moulds may be bent and scribed. It will be seen that the face moulds may be developed on the elevation, but a separate diagram is introduced here to save the confusion of lines.

Referring again to the radial joint surfaces, it is evident that the straight lines CD, EF, etc., Fig. 263, become helices when folded around the cylindrical surfaces, and would be curved in elevation. Plane joints may be obtained, but since they would intersect the upright cylindrical surfaces in elliptic curves, the joint lines between the developed face moulds would be slightly curved. Plane joints would, however, project as straight lines in a direct elevation, since it would give an edge view of each plane, and by developing intermediate points in the joint arrises the joint curves in the face moulds are easily obtained. There is little, if any, advantage in having plane joint

surfaces, and the additional work involved in setting out to obtain them is hardly justified.

Circle-on-Circle Masonry Arch: The Soffit Radiating at the Springing and Horizontal at the Crown.

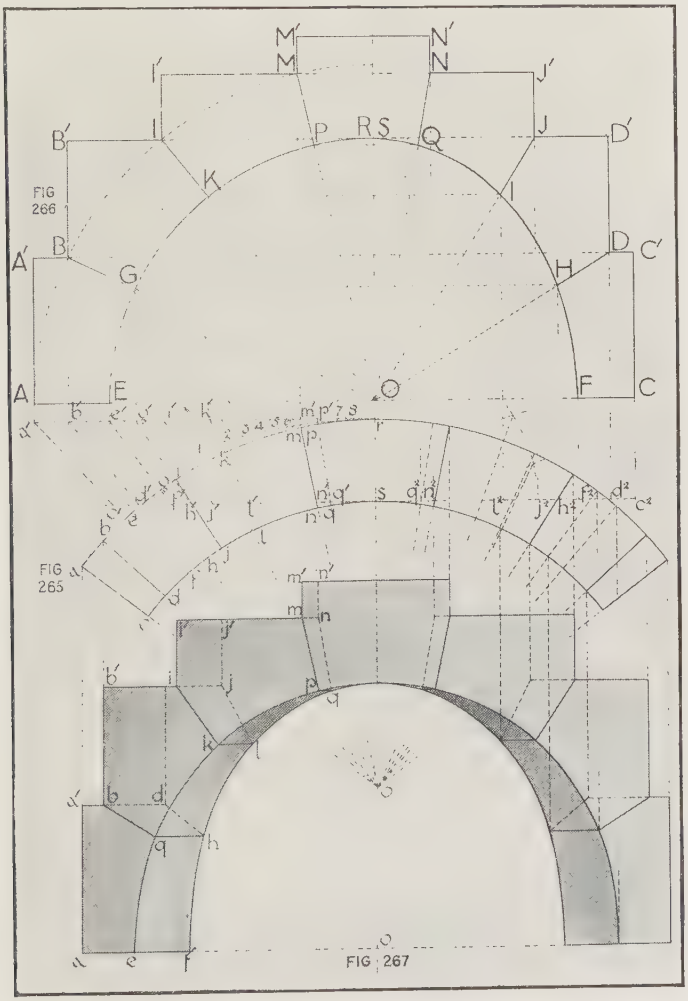
Figs. 265, 266, *and* 267.—The inverted plan of the arch is shown in Fig. 265. It is required that the development of the soffit arris on the convex side shall be a semicircle, and the joint lines across the soffit shall be horizontal lines, radiating in plan. Draw a^1r tangent to the outer plan segment at r. Let the radiating lines ac and ef be the straight arrises on the base of the springer, arranged to give the required span. Develop the arc ra into the line a^1r, so that a^1r equals the length of the arc ra and e^1r equals the length of the arc er. About any point O in or produced draw the quadrant of a circle ER, Fig. 266, and a parallel quadrant from A; A and E being projected from a^1 and e^1 respectively. Draw radiating lines GB, KI, and PM between the quadrants to represent the developed joint lines between the required voussoirs. Draw upright and horizontal lines from each of the points B, I, M, an upright line from A, and a horizontal from M^1 at any required height. Then the upright and horizontal lines represent the development of the stepped arrises on the outer face of the arch.

Project G, K, and P and B, I, M on to the straight line ra^1 at $g^1k^1p^1$ and $b^1i^1m^1$ respectively, and by the process explained in Chapter I., problem 6, turn b^1 to b, g^1 to g, i^1 to i, k^1 to k, m^1 to m, and p^1 to p.

Note.—Centre 2 of the arc bb^1 is on the perpendicular bisector of b^1r, and on the arc which is drawn tangent to r, radius one and a half times ro; o being the centre of the plan curves. All the required centres are readily determined in a similar way.

Draw radiating lines across the plan from points b, g, i, k, m, and p to meet the inner plan segment in points d, h, j, l, n, and q respectively. Draw a tangent to the inner plan segment at s, and draw radiating lines from all the points in a^1r to meet it in c^1, d^1, f^1, h^1, j^1, l^1, n^1, and q^1; then sc^1 is the stretch-out length of the arc sc, sd^1 is the length of the arc sd, and so on to sq^1 equal to the arc sq. Transfer the points in sc^1 into the tangent produced on the opposite side of s, making sc^2 equal sc^1, sd^2 equal sd^1, and so on to sq^2 equal sq^1. Project q^2 to meet a horizontal from P in Q, n^2 to meet a horizontal from M in N, and similarly for the remaining points, the appropriate horizontals being found by reference to the plan radials on the left of the centre line; thus because bd is a radial in plan, BD is horizontal in development; gh is a plan radial, GH is horizontal. It will now be found that S, Q, L, H, and F

are points in an elliptic quadrant, the curve being the development of the inner arris of the soffit. When the lines are drawn between the points, Fig. 266, as shown, the right-hand half of the figure is the



Figs. 265, 266, and 267.

development of half the concave face of the arch, and the two developments provide the face moulds required as in the preceding example. The joint lines on the inner face developed radiate to *O*, and are not, therefore, normal to the elliptic curve.

The joint surfaces are twisted, and the joint arrises are helical. The bed moulds are the plan outlines of the voussoirs, as follows ; *aghc* for the springers, *bdkl* for the second voussoirs, *ijpq* for the third voussoirs, and *mnrs* is half the keystone bed mould.

An elevation of the arch is shown in Fig. 267, the slightly curved joint lines across the faces being drawn straight.

The Cuneoid.

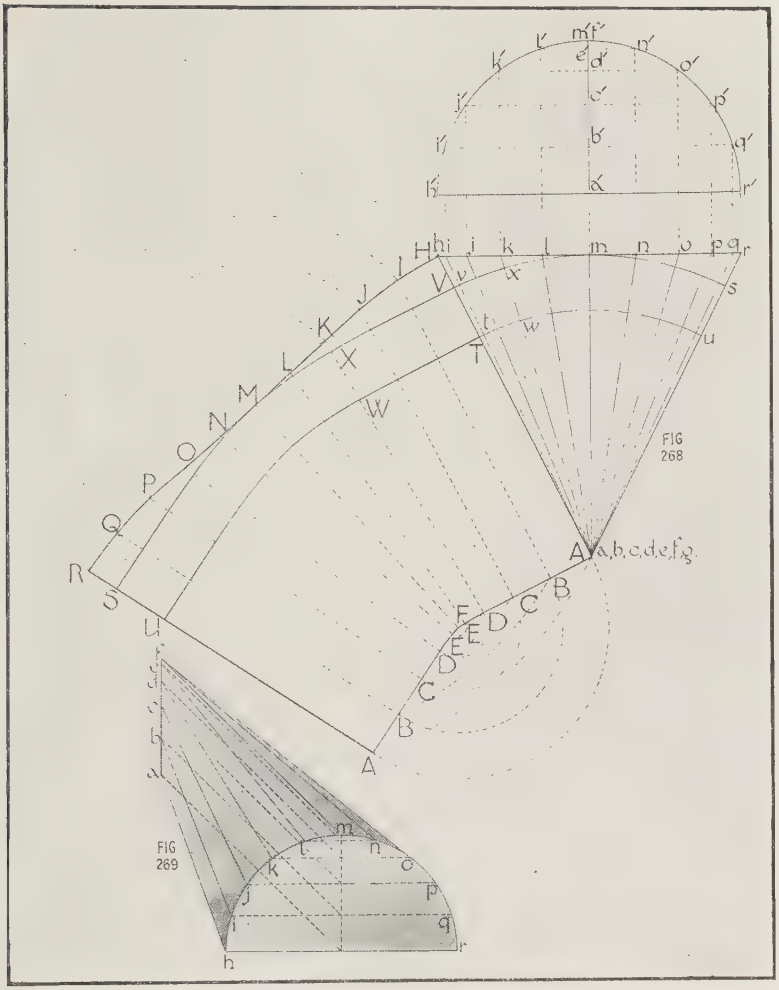
Fig. 268.—In setting out door and window frames which are circular in both plan and elevation and have radiating jambs, there are several advantages to be obtained from introducing the solid known as the " cuneoid ".

In Fig. 268 the triangle *ahr* is the plan, the semicircle $h^1m^1r^1$ the elevation of the solid, whose nature will be best understood after considering the pictorial view in Fig. 269. Referring to the latter figure, let *hmr* be a semicircle whose plane is vertical, and let *mf* be a straight line perpendicular to this plane. Let *af* be a vertical straight line having a length equal to the radius of the semicircle. Let *bi*, *cj*, *dk*, and *el* be any horizontal straight lines intersecting the line *af* and the semicircle ; then these lines all lie in the surface of the solid. It will be seen that any horizontal section of the solid is a plane isosceles triangle having its vertex in the line *af* and its base in the plane of the semicircle. In the plan and elevation, Fig. 268, the lettering corresponds to that in Fig. 269.

The surface of the solid is not developable, but the approximate development AFA^1RMH is sufficiently accurate for practical purposes, and is obtained as follows : Draw AB at right angles to ah and equal to a^1b^1. With the length of the arc h^1i^1 as radius, and centre h, draw a circular arc, and with radius ai, centre B, cut this arc in I ; join BI. Draw CB at right angles to BI and equal to b^1c^1. With the length of the arc i^1j^1 as radius, centre I, draw a circular arc, and with aj as radius, C as centre, cut it in J ; join JI. Proceed in a similar manner to MF, and, since MF is the axis of symmetry, the other half of the development may be obtained by intersecting arcs, centres M and F, as shown in the figure.

The student will understand the method of development by considering each section of the area of the solid independently. Thus, the area *abhi* in plan, $a^1b^1h^1i^1$ in elevation, is bounded by three straight lines and an arc. Two of the straight lines, AH and BI, are horizontal, and therefore their true lengths are seen in plan. The third straight line, AB, is vertical, and its true length is seen in elevation. The arc is

parallel to the vertical plane and its length is seen in elevation. Before development, *AH* and *BI* are perpendicular to *AB*, but cannot both remain so in development, since the width of the curved surface about



Figs. 268 and 269.

the arc must be greater than the straight width at *AB*. If the angles at *A* and *B* were made equal, the approximation might be nearer, but this would involve more work than the result would justify. Usually, *AF* is made straight and equal to a^1f^1, but the method explained is equally simple and there is less error. Each section may be considered

in the same manner as *ABHI*, and other equally suitable methods may probably be evolved.

When the soffit of a circle-on-circle frame is cuneoidal, the part of the surface of the cuneoid involved lies between two curves whose plans are circular arcs similar to *vs* and *tu*, both of centre *a*. The development of the arcs is readily obtained on the developed surface of solid, since all the straight horizontal lines lying in the surface of the solid between each arc and the line *af* are of equal length; therefore, if the radius of the arc be set out along each line, *AH*, *BI*, *CJ*, etc., as *DW* equals *dw*, and *DX* equals *dx*, smooth curves through the points are the required developments.

Circle-on-Circle Solid Frame with Radiating Jambs, in Two Segments.

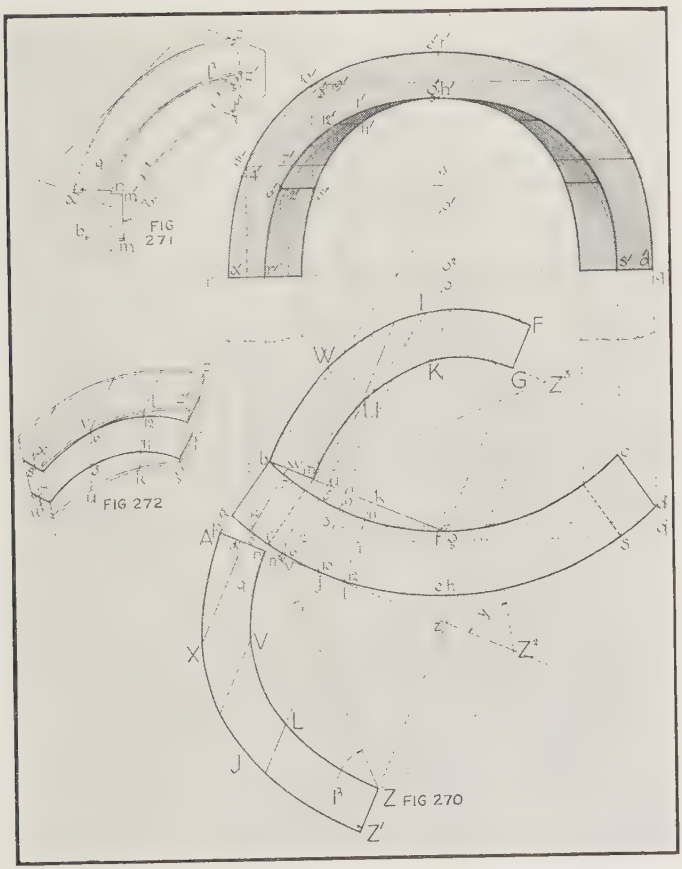
Figs. 270, 271, *and* 272.—In Fig. 270, *abcd* is the plan outline and $a^1e^1d^1$ the elevation outline of a solid frame having its inner and outer faces in the surfaces of a vertical hollow cylinder and its soffit and crown in the surface of two cuneoids.

The plan curves *ad*, *bc* are drawn from the centre *o*, *o* being the plan of the straight line vertex of the soffit cuneoid, and the soffit springing lines from *r* and *s* converge towards *o*. *ab* and *cd* are parallel to *ro* and *so*, and converge towards o^1; o^1 being the plan of the straight-line vertex of the crown cuneoid.

To draw the projections. Draw the plan curves to any given radius as *og*, and *oh*. Draw *os* and *or* to represent the springing arrises of the soffit. Make *ra* and *sd* each equal to the required thickness of the frame, and draw *ab* and *dc* parallel to *ro* and *so* respectively to represent the outer springing arrises. Produce *ab* and *dc* to meet in o^1. Join *rs*, and project *r* and *s* to r^1 and s^1. Draw a semicircle (shown dotted) on r^1s^1 to cut the centre line in g^1. Make g^1e^1 equal to *sd* and draw about the same centre, o^2, a second semicircle to meet the springing line r^1s^1 produced in p^1 and q^1; this semicircle is also shown dotted. Project p^1 and q^1 to meet *ab* produced in *p*, and *cd* produced in *q* respectively. Join *pq*. Then *ros*, $r^1g^1s^1$ are the projections of the soffit or inner cuneoid, and pqo^1, $p^1e^1q^1$ are the projections of the outer cuneoid, and all straight lines from *o* across the soffit are horizontal, and all straight lines from o^1 across the back of the frame are horizontal.

The elevation is obtained by projecting any number of points in the semicircle r^1g^1 into the straight line *rs*, and drawing the projections of straight lines through the points across the soffit, and similarly for the back of the frame; thus, project 1^1 and 2^1 to meet *rs* in 1 and 2, and

3^1 and 4^1 to meet *pq* in 3 and 4. Through 1 and 2 draw straight lines in the direction to pass through *O* to meet the plan curves in 11, 12, 5, 6. Project 11 and 12 to meet a horizontal through 1^1 in 11^1 and 12^1, and 5 and 6 to meet a horizontal through 2^1 in 5^1 and 6^1. Then 5^1, 6^1, 11^1, and 12^1 are points in the elevations of the soffit arrises. Project 3^1 and 4^1 to



Figs. 270, 271, and 272.*

3 and 4 in *pq*, and through 3 and 4 draw straight lines converging towards O^1, and meeting the plan curves in points 7, 8, 9, and 10. Project 7 and 8 to meet a horizontal through 4^1 in 7^1 and 8^1, and 9 and 10 to meet a horizontal through 3^1 in 9^1 and 10^1. Then 7^1, 8^1, 9^1, and 10^1 are points in the outside arrises. The curve through $e^1 7^1$ and 9^1 is not drawn.

To obtain the face moulds required in working the head.

Join bg and draw the tangent Az parallel to bg. The distance between bg and Az is the required thickness of the material to contain one section of the head. The face moulds are sections through the two cuneoids by vertical planes containing bg and Az, these planes being the faces of the plank from which the section is to be taken.

Produce the numbered lines across the plan to meet the face planes bg and Az in points w, u, i, k, x, v, j, and l; produce the soffit springing line to m and n, and the centre line ge to z. Draw perpendiculars to bg from points w, u, i, k, and g, and perpendiculars to Az from points x, v, j, l, and z. Set out on each perpendicular the height of the horizontal line from whose extremity it is drawn; thus, the perpendiculars from k and l intersect the horizontal line through l; then kK and lL are each made equal to the height of l^1 above the springing line r^1s^1. gG equals o^2g^1, fF equals o^2f^1, and similarly for the remaining points. Then $bmGF$ and $AnZZ^1$ are the required face moulds. Draw mm^1 perpendicular to bg and join m^1Z; then m^1Z is the position of the edge of the plank relative to the mould $AnZZ^1$. Produce Zz to meet a perpendicular to fF from G in Z^3; then the straight line mZ^3 is the position of the edge of the plank relative to the mould $bmGF$.

The face bevels for the joint planes are shown at α and β. The lower edge bevel is a right angle. The upper edge bevel is obtained as follows: Produce m^1m to meet the centre line in y. With m^1 as centre, turn Z to Z^2 (in Az produced) and join Z^2y; then γ is the required bevel. For, the plan of the line of intersection between the joint plane $EFGH$ and the plane of the edge of the plank is in the centre line zf. And m^1m is a horizontal line in the plane of the edge; therefore the two lines produced meet in y. Since the stock of the bevel is applied along the arris between the face and edge of the plank, and the blade is required to lie along the line of intersection, the required angle is that between the arris and the line of intersection, and since these two lines and the line m^1y form a plane right-angled triangle (zm^1 being at right angles to m^1y, the latter line being horizontal), the right-angled sides being equal to m^1y and m^1Z, the true shape of the triangle is m^1Z^2y, and γ is the angle between the arris and the line of intersection between the edge and joint planes.

The moulds applied to the material are shown in the sketch, Fig. 271, the points and angles being lettered to correspond with Fig. 270.

Fig. 272 shows the method of determining the plan curves on the material after it has been worked to the face moulds, and the letters and numbers show the distances to be taken from the plan and set off

along the level ordinates, *kl*, *vu*, and *wx*. Neither of the Figs. 271, 272 are to scale, and both are distorted views.

The geometry of circle-on-circle frames made with three or more sections in the circular head is rather more involved than is the preceding example. In joinery, plane joints must be arranged, even at the expense of more complicated setting out, owing to the extreme difficulty of making perfect joints between winding surfaces.

In all circular work the joint planes should be arranged normal to the centre line curve, or as nearly normal as the nature of the problem will admit, so as to avoid unduly sharp arrises and keep the surfaces at somewhere near right angles to the pressures on the joint. To obtain these results the joint arrises are frequently curved lines, and some uncertainty may exist as to how far these curves may be ignored and straight lines substituted.

An analysis of the curved joint arrises in a solid head, circular in both plan and elevation, made in three sections with parallel jambs, is illustrated in Fig. 273, and should prove helpful in some following examples.

It has already been explained that the head of a circle-on-circle frame with parallel jambs is a solid formed by the interpenetration of a hollow vertical cylinder and a hollow horizontal cylinder. In Fig. 273 two such cylinders are shown. Point *o* is the plan of the axis of the vertical cylinder, and o^1 is the elevation of the axis of the horizontal cylinder. Let *e* and e^1 be the projections of any point in the line of intersection of the centre layers of the two cylinders. Draw o^1g^1 through e^1, and o^2g through *e* tangent to the centre layer of the vertical cylinder. Draw a straight line from *o* through *e* to meet *XY* in *s*. Through o^2 draw *ht* parallel to *os*. Draw a perpendicular to *XY* from *s* to meet a horizontal from e^1 in s^1. Draw *vt* through s^1. Let *vth* be a section plane through the two cylinders; then if g^1 and f^1 in the normal o^1e^1 be projected to *g* and *f* in the tangent o^2e, gg^1 and ff^1 are projections of points in the outlines of the sections of the cylinders by the plane, and any number of points in the section outlines may be obtained in a similar manner. Thus, through any point *m* in *ht* draw a straight line parallel to o^2e to cut the plan circles of the vertical cylinder in points 22 and 30. Project *m* to m^1 in *XY*, and through m^1 draw a parallel to o^1e^1 to meet the elevation circles of the horizontal cylinder in 11^1 and 3^1. Project 11^1 and 3^1 to 11 and 3 in the line 30*m* produced, and produce 3^1m^1 to meet perpendiculars from 22 and 30 in 22^1 and 30^1. Then 3 and 11 are points in the plan of the section through the horizontal cylinder, and 22^1 and 30^1 are the elevations of

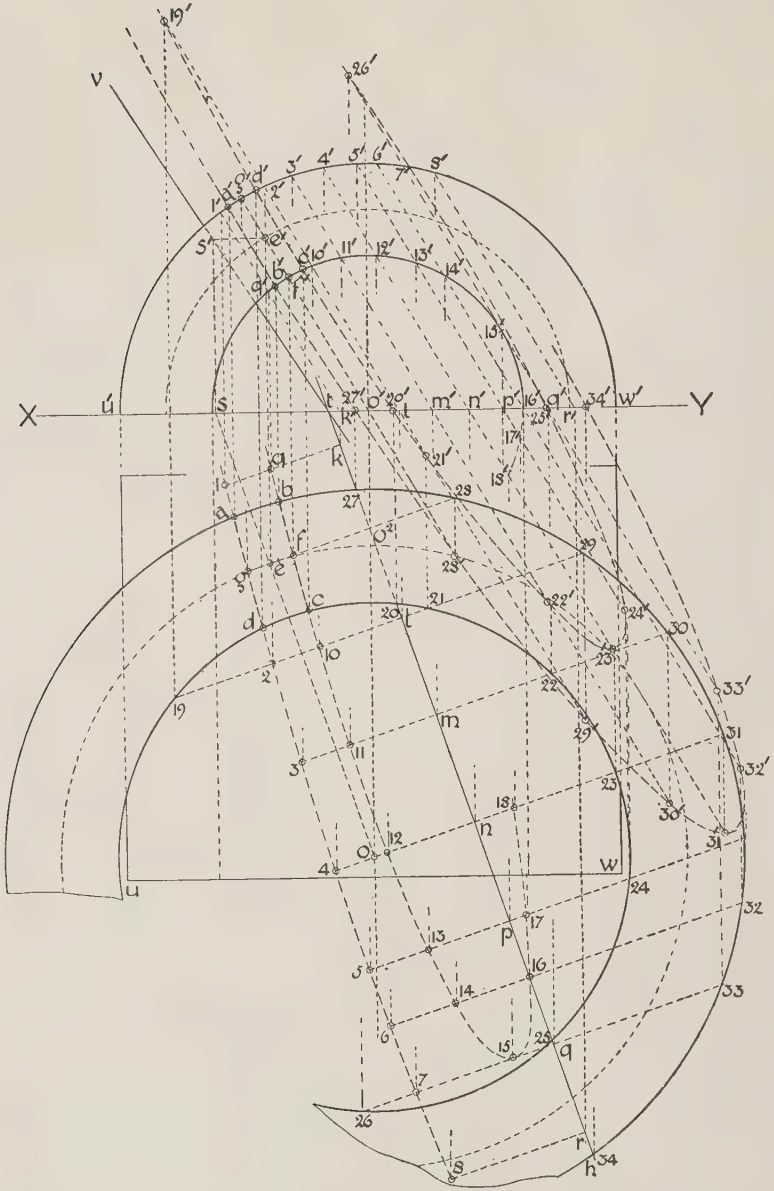


Fig. 273.*

points in the section through the vertical cylinder. For 3-30 and 3^1-30^1 are the projections of a straight line in the plane *vth*; therefore the points in which the line intersects the surfaces of the cylinders are points in the lines of intersection between the plane and cylinders. Other lines in the plane parallel to *OE* are shown by their projections, intersecting *ht* at *k*, *l*, *n*, *p*, *q*, *r*, and intersecting *XY* at $k^1l^1n^1p^1q^1r^1$, and points 1, 2 . . . 8; 9, 10 . . . 18; 19^1, 20^1 . . . 26^1; 27^1, 28^1 . . . 34^1 are points in the projections of the elliptic section outlines, all obtained precisely as explained for 3, 11, 22^1, and 30^1. The plan curves 1 . . . 8, 9 . . . 18 intersect the plan circles in *a*, *d*, *b*, and *c*, and it will be seen that the elliptic arcs *agd* and *bef* are of very slight curvature. Project *a* to a^1, *b* to b^1, *c* to c^1, and *d* to d^1; then $a^1b^1c^1$ and d^1 are in the elevations of the elliptic section outlines of the vertical cylinder, and again the arcs a^1b^1, d^1c^1 are of very slight curvature.

The area common to both sections is bounded by the four slightly curved elliptic arcs *AB*, *BC*, *CD*, *DA*, and in the following example the joint plane must be regarded as a plane figure, similar to *ABCD* but having straight lines substituted for the elliptic arcs. It should be remembered, however, that this device does not affect the final result, since the elliptic curves are produced in the completed head, and the straight lines are introduced simply to avoid unnecessary complexity in setting out. This point will be referred to again, but Fig. 273 should be carefully studied before considering the following examples.

Moulds and Bevels for the Head of a Solid Frame, Semicircular in Elevation, Segmental in Plan, with Parallel Jambs; the Head to be in Three Sections.

Fig. 274.—The plan segments are drawn about *o* as centre, the elevation semicircles about o^1 as centre. Let *k* and k^1 be the projections of the point in which the centre line is required to intersect the joint plane. Draw the normal k^1o^1, and draw o^2k tangent to the plan centre line, o^2 being in the centre line oo^1. Join *ok*, and produce to *l*, and project *l* to meet a horizontal from k^1 in l^1. Draw the horizontal trace of the joint plane through o^2 parallel to *ol*, and the vertical trace through l^1. Parallel to o^2k draw tangents to the plan curves to meet the horizontal trace in *r* and *s*. Project *r* and *s* on to a^1d^1 at r^1 and s^1. Draw $s^1h^1j^1$ and $r^1g^1i^1$ parallel to o^1k^1. Project i^1, j^1, h^1, and g^1 to the plan tangents at *i*, *j*, *h*, and *g*. Join *ij* and *gh*. Then the straight lines *ij* and *gh* in plan, and j^1h^1, i^1g^1 in elevation, represent the slightly curved elliptic arcs discussed in the preceding article, and may represent the section of the head by the joint plane.

Draw the chord ag, and parallel to it draw a tangent to the outer plan segment. Produce ij and gh to meet the chord in m and the tangent in n and p, and produce the springing line of the soffit to meet the chord in e and the tangent in f. Draw the projections of any desired number of level ordinates across the soffit, as 3-4, 3^1-4^1, and also across the upper surfaces, as 1-2, 1^1-2^1. Draw perpendiculars to ag from 1, 2, f, 3, 4, m, n, g, and h, and on these perpendiculars set out

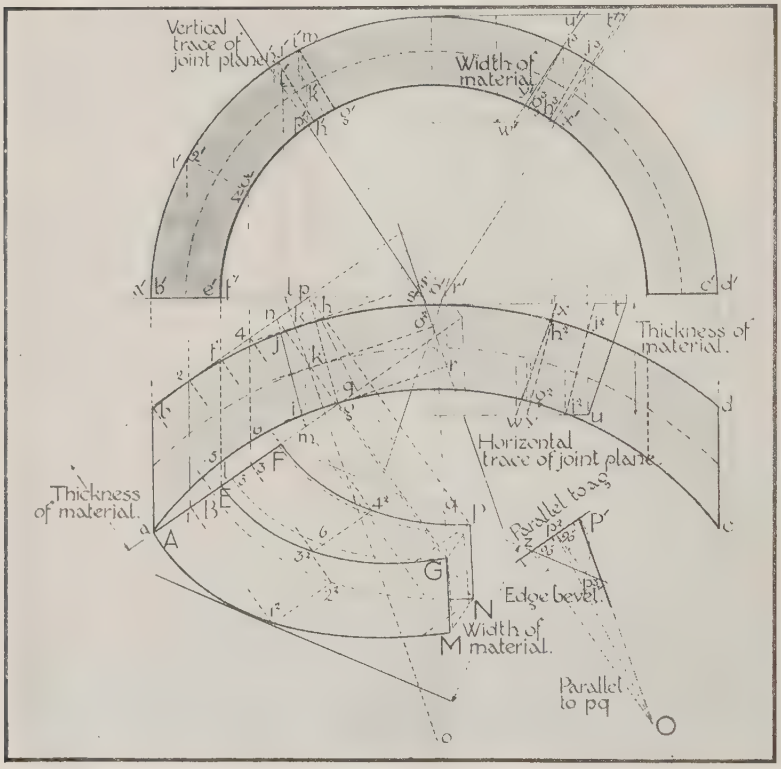


Fig. 274.*

distances from ag equal to the heights of the various points above a^1d^1; thus, 1-1^2 equals the height of 1^1 above a^1d^1, and gG equals the height of g^1 above a^1d^1; the heights of m, n, and p are found by projectors from these points to cut the elevation curves in m^1, n^1, and p^1 respectively. Draw curves through A, 1^2, M; B, 2^2, N; E, 3^2, G; and F, 4^2, P. Join PN and GM. Then $AEGM$ is one required face mould, and $BFPN$ the second required face mould, and the positions in the figure represent the positions of the moulds when applied on the

opposite faces of the plank, relative to the edge of the plank which is indicated by the straight line through F and P. The plan curves are shown dotted around the soffit $EGPF$, and on the joint plane $GPMN$.

It should be noted that although the joint arrises on the finished frame are elliptic, as explained in connection with Fig. 273, the joint lines GM and PN across the face moulds are straight lines, since they are the lines of intersection between the joint plane and the vertical face planes of the material. In working the section, the soffit and upper surface must be made straight at every position, in the direction parallel to the level ordinates 3^2-4^2 and 1^2-2^2; then the joint arrises between G, P and M, N will become elliptic in the process of working. Similarly, if the vertical cylindrical faces are worked straight at every position parallel to the vertical ordinates 1-1^2 and 3-3^2, the remaining two joint arrises will become elliptic. Since GM and PN are the lines of intersection between the joint plane and the vertical face planes, MG produced will meet ag produced in the horizontal trace of the joint plane, and NP produced will meet ag produced in the projection of the point of intersection between the tangent fp and horizontal trace, the projection being made perpendicular to ag.

The face bevels for preparing the joint planes are the angles AFP (for the lower joint, $ABEF$) and FPN (for the upper joint, $GMPN$). The edge bevel for the lower joint is a right angle, the edge bevel for the upper joint being determined as follows: Through any point z in FP produced draw a straight line parallel to ag, and from z draw a perpendicular zO to ag. From any point p^2 in the parallel draw p^2O parallel to pq (pq is the plan of the line of intersection between the joint plane and edge plane of the material, q being projected from q^1 in FP and MG produced). Draw p^2p^3 perpendicular to ag, and with z as centre, zp^3 as radius, cut zp^2 produced in P^1. Join P^1O. Then zP^1O is the required angle. This bevel may be found by using existing lines in the plan, as explained in the previous examples, but the supplementary drawing is introduced to avoid confusion. The bevel is actually the angle between the lower arris of the material and the line of intersection between the lower edge of the material and the joint plane, and any suitable construction may be used to determine it.

The moulds, bevels and material dimensions for the centre section of the head are shown on the right of the centre line. Let $g^2h^2i^2j^2$, $g^3h^3i^3j^3$ be the projections of the joint surface, determined as before. Draw a straight line uw through i^2 perpendicular to the centre line oo^1, and draw the tangent to the outer plan curve parallel to uw. Produce g^2h^2 to meet uw in y, and the tangent in x. Draw wx parallel to the

horizontal trace of the joint plane, and project x to cut the lower elevation curve in x^1. Through x^1 draw a perpendicular to the centre line and project w to w^1. Draw x^1t^1 and w^1i^1 parallel to the vertical trace of the joint plane and produce w^1i^1 to meet the crown tangent in u^1 (t^1 is also in the tangent). Project u^1 to u and t^1 to t, and join tu. Then the elevation curves j^1j^3, h^1h^3 and the straight lines j^1h^1, j^3h^3 are the outlines of one face mould, and i^1i^3, g^1g^3, i^1g^1, i^3g^3 the outlines of the second face mould. The face and edge bevels for forming the joint are the angles $x^1t^1u^1$ and xtu respectively. The positions of the moulds when applied to the material are as shown in elevation. The plan curves are determined in the manner already explained.

Circle-on-Circle Solid Frame with Radiating Jambs in Three Segments.

Fig. 275.—The head of the frame is arranged with its soffit in the surface of the cuneoid omn in plan, and m^1n^1 is the semicircular vertical section of the solid. The upper surface of the frame is in the surface of a second cuneoid, the plan of whose straight line vertex is o^1, the vertical semicircular section pq being shown in elevation at p^1q^1.

Divide the semicircle p^1q^1 into three equal arcs to obtain a^1, and draw the radius a^1c^1 cutting the inner semicircle in b^1. Project a^1 to a in pq, and b^1 to b in mn. Draw a straight line through a and b to meet the centre line in c. Draw fg through b in the direction of o, and de through a in the direction of o^1. Produce ed and gf to meet the springing line p^1q^1 in k and l respectively, and draw perpendiculars from k and l to meet horizontals through a^1 and b^1 in k^1 and l^1 respectively; then the horizontal trace of the plane of the joint passes through c and is parallel to ak and bl, and the vertical trace is through k^1 and l^1. Project d and e to d^1 and e^1 on the horizontal through k^1 and a^1, and g and f to g^1 and f^1 in the horizontal through l^1 and b^1. Join d^1, f^1 and e^1, g^1; then $defg$ and $d^1e^1f^1g^1$ are the projections of the section of the head by the joint plane. Draw the chord rg and parallel to it draw the tangent to the outer plan curve; then the chord and the tangent represent the vertical plane faces of the material, the distance between them being the thickness of the material required. Project any number of points in the elevation circle m^1b^1 on to mb, the plan of the circle, as 1^1 to 1, and draw the plans of level ordinates through the plans of the points to meet the face planes (only one such point, 1, is shown in the figure); thus, the level ordinate wx through 1, being in the soffit, is directed towards o. Similarly, yz is the plan of a level ordinate in the upper cuneoidal surface, 2 and 2^1 being the projections of the point

in which the ordinate intersects the circular section of the surface; yz is of course directed towards o^1. Produce gf, ed, and mu to meet the face planes in j, h, and i and t respectively, and erect perpendiculars to the face planes from j, i, g, h, w, x, y, and z, setting the appropriate heights, taken from the elevation, along the perpendiculars to obtain the face moulds; thus jJ and gG are each equal to the height of b^1;

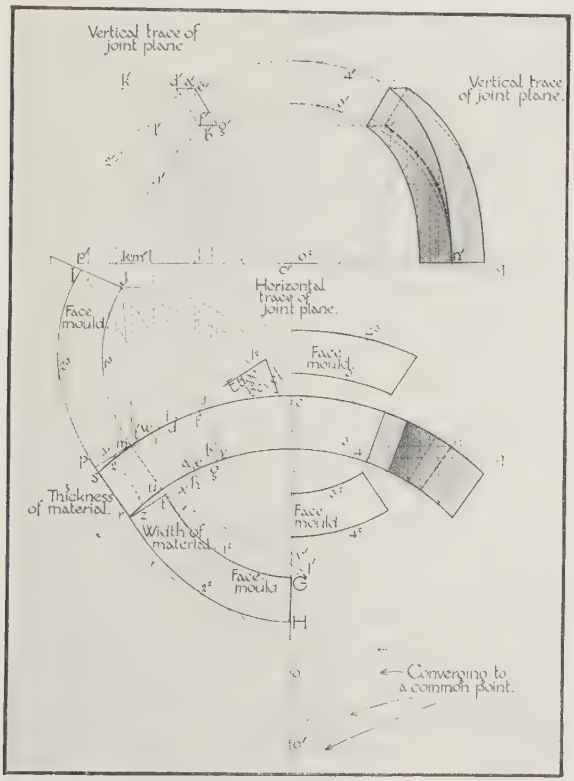


FIG. 275.

iI and hH are each equal to the height of a^1; $x1^2$ and $w1^3$ each equal the height of 1^1; and $z2^2$ and $y2^3$ each equal the height of 2^1.

NOTE.—The joint lines IJ and HG produced meet the face planes produced on the horizontal trace.

Project t perpendicularly through the material to t^1, and join t^1J. Produce Jj to J^1 (GJ^1 being parallel to rg) and join t^1J. Then t^1J and tJ^1 are the positions of the edges of the plank relative to the applied face moulds, and rtJ^1 and t^1JI are the face bevels for the joints. To

obtain the edge bevel, project v^1 (in tJ^1 and HG produced) to v in rg produced. Join vj; then vj is the plan of the line of intersection between the lower edge of the material and the joint plane, and the required bevel is the angle between vj and the line, or arris, of intersection between the face and edge of the material. With t^1 as centre, turn J to J^2. Produce t^1t and jv to meet in O (not shown in the figure) and join J^2O; then jJ^2O is the required angle.

NOTE.—The horizontal trace of the joint plane also passes through O, and may be used in finding the edge bevel, allowing jv to be dispensed with.

The moulds, bevels, and the dimensions of material for the centre section are determined on the right of the centre line. The construction is similar to that explained in the preceding example, and no further explanation should be necessary.

In circle-on-circle constructions comprising a masonry arch, a frame, and a sash or door, the positions of the semicircular cuneoidal sections should be given to the mason and the joiner to ensure a reasonable joint being made between the arch and frame. This would obviate the necessity for taking a mould from the finished arch from which to design the frame, and would give a more satisfactory result.

PLATE VIII.

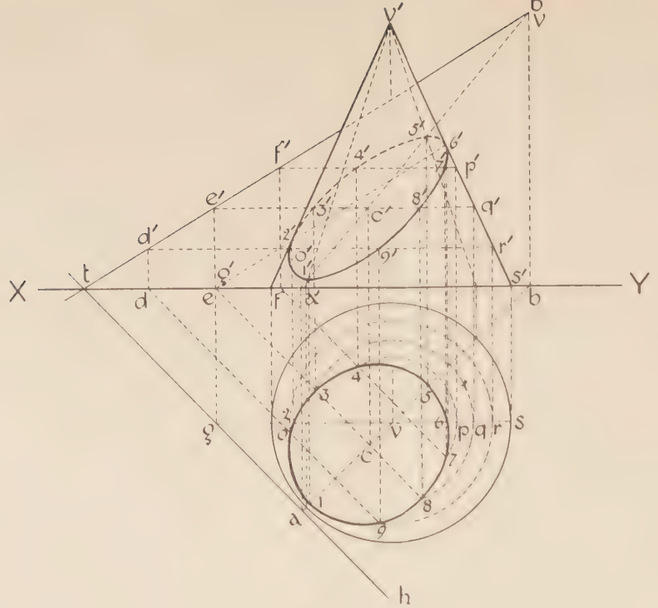
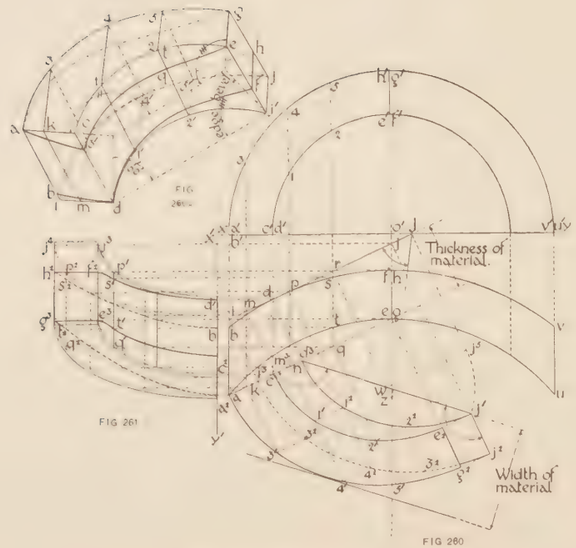
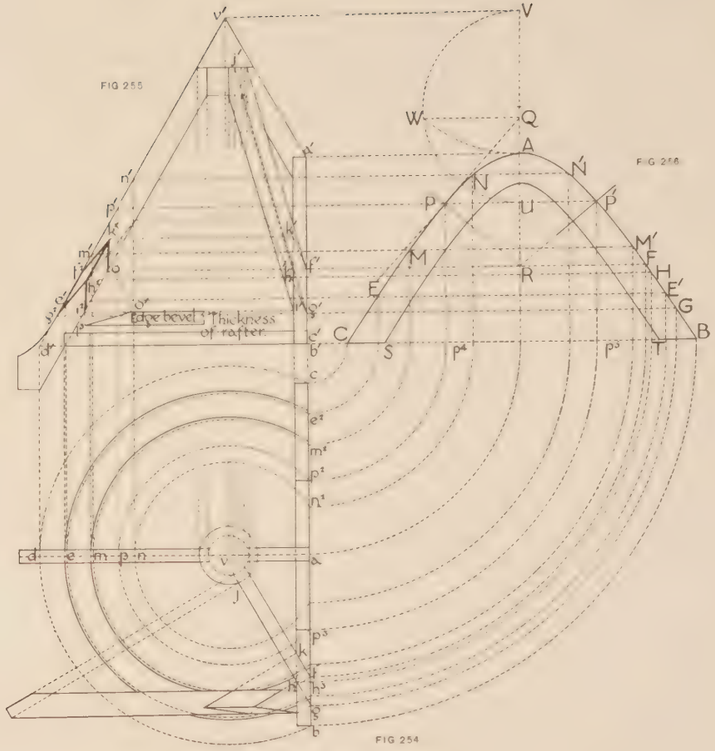


FIG. 250.*



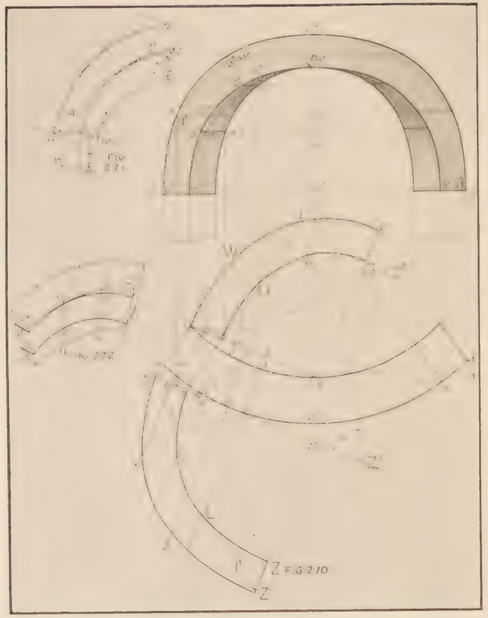
FIGS. 260, 260*a*, and 261.*



FIGS. 254, 255, and 256.*

PLATE IX.

Follow Plate VIII.



FIGS. 270, 271, and 272.*

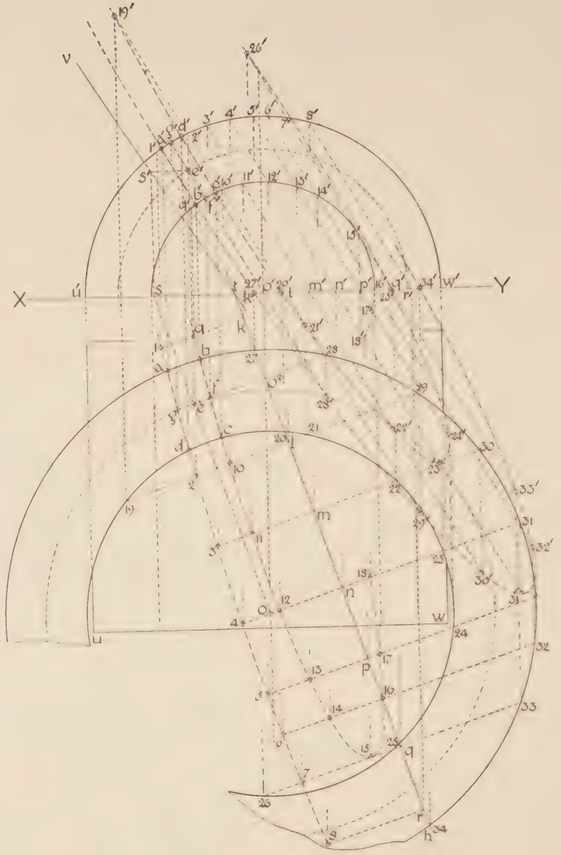


FIG. 273.*

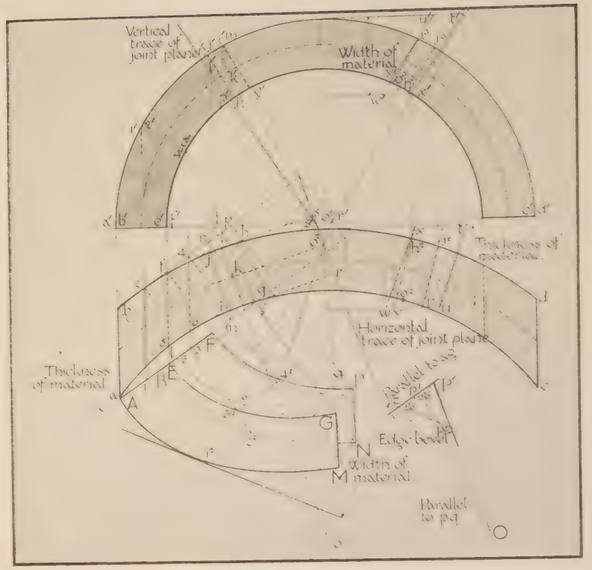


FIG. 274.*

CHAPTER XVI

THE GEOMETRY OF WREATHED HANDRAILS

In the tangent system of producing handrail wreaths the geometrical processes are: to obtain the plane sections of a hollow cylinder by two parallel oblique or inclined planes, and to determine the angles between these planes and two vertical planes tangent to the surface of the cylinder. The sections of cylinder give the "face moulds" and the angles between the planes give the "twist bevels".

The Theory of Handrailing.

Fig. 276.—Let the dotted arc *ac*, centre *o*, be the plan of the centre line of a quarter-turn wreath, and let abb^1 and bb^1cc^1 be vertical planes tangent to the centre line plan at *a* and *c*. If ab^1 and b^1c^1, the upper edges of the tangent planes, be inclined at the same angles to the ground as are the straight sections of the handrail connected by the wreath, the projection of the arc *ac* on to the plane containing ab^1 and b^1c^1 is an elliptic curve, and is the true shape of the centre line of the wreath.

Continue *cb* and c^1b^1 to meet in *d*, and draw the straight line *da*. Then if *ac* be regarded as lying in the horizontal plane, and c^1b^1 in the vertical plane, *da* and dc^1 are the traces of the oblique plane containing ab^1 and b^1c^1. Let o^1 be the point in the oblique plane directly above *o*, and let o^1g^1 be parallel to *ad*; then o^1g^1 is a horizontal line whose plan is *og*, and since *o* is the centre of the circular plan, o^1 is the centre of the elliptic projection, and because o^12^1 equals *o*2, the radius of the plan circle, o^12^1 is the minor axis of the elliptic arc. Let *ef* and *mn* be any straight lines across the plan parallel to *da*, cutting the plan curve in points 1 and 3 respectively; then if *f*, *e*, *m*, and *n* be projected vertically into the oblique plane at f^1, e^1, m^1, n^1, the straight line f^1e^1 will cut the elliptic arc in 1^1, the vertical projection of 1, and the straight line m^1n^1 will cut the ellipse in 3^1, the vertical projection of 3. The planes aoo^1 and oo^1cc^1 are the springing planes.

The dihedral angle between the oblique plane adc^1 and the vertical tangent plane bb^1cc^1 is called the upper twist bevel, and between the planes adc^1 and abb^1, the lower twist bevel. The figure is a distorted view only, but the method of obtaining the twist bevels is illustrated at apb and dqb, where bp equals the perpendicular distance of b from ab^1.

In a true geometric figure, α would be the upper bevel and β the lower bevel. The upper edges of the tangent planes ab^1 and b^1c^1 are called the " pitches " of the straight rails connected by the wreath.

Now consider Fig. 277. Let the quarter circles represented by the arcs sm and rn be the outlines of the plan of a quarter-turn wreath,

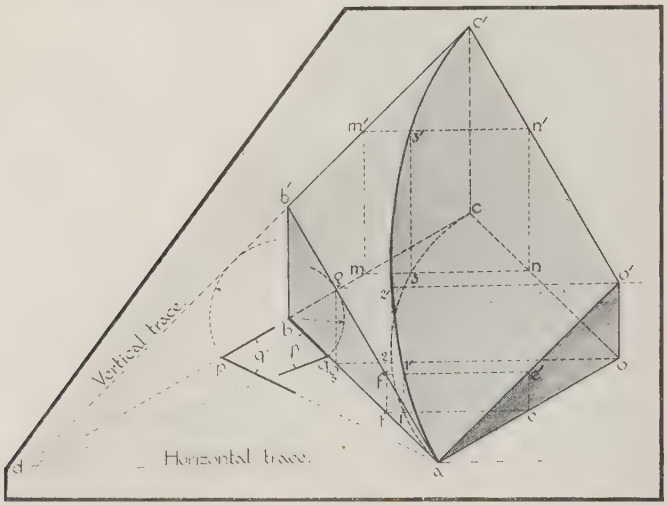


Fig. 276.*

and let ac be the plan centre line. Let o be the centre of the plan curves, ab and bc the plans of the vertical tangent planes, oa and oc the plans of the springing planes. Now regard $a^1b^2c^2o^2$ as the plan square $abco$ raised through the vertical height aa^1, and let a^1b^1 and b^1c^1 represent the upper edges of the tangent planes. Then, as in the preceding figure, da is the horizontal trace of the plane containing a^1b^1 and b^1c^1 and the elliptic arcs s^1m^1 and r^1n^1 are the vertical projections of the curves sm and rn determined by horizontal ordinates, as before.

Let m^2m^3, n^2n^3, r^2r^3, and s^2s^3 be perpendiculars through m, n, r, and s respectively, and each equal to the vertical thickness of the inclined plank from which the wreath is to be taken, and let j^2j^3, k^2k^3, etc., be similar lines through any other points in the elliptic arcs r^1n^1,

s^1m^1; then the curves through $n^3j^3r^3$, $m^3k^3s^3$, $n^2j^2r^2$, $m^2k^2s^2$ represent the face moulds applied to the material in their proper relative positions, and it will be seen that the solid lying between the moulds stands directly over the plan of the wreath. Let the straight lines s^3q, s^2p, r^3u, and r^2t be parallel to a^1b^1, and the plane $qtpu$ be perpendicular to a^1b^1, then the part of the wreath between the springing plane $r^2r^3s^2s^3$ and the joint plane $pqtu$ will be straight, and this straight portion is

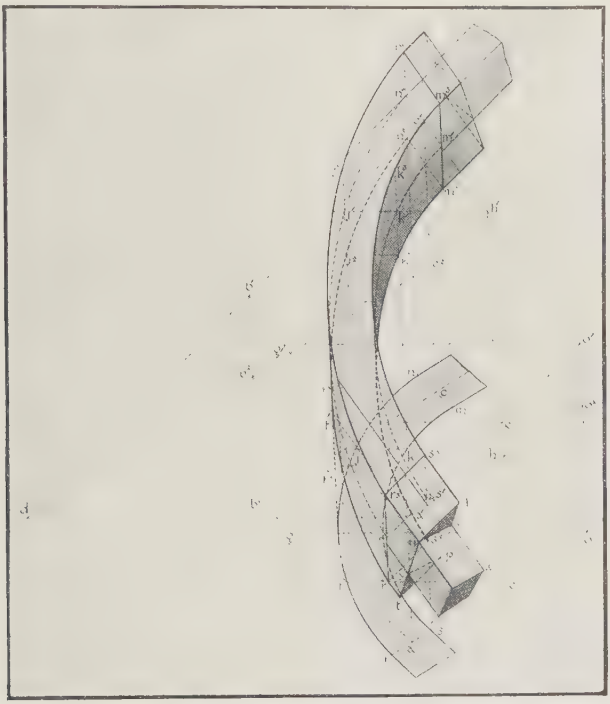


Fig. 277.

called the shank. Short pieces of the two straight rails are shown intersecting the solid lying between the face moulds, and the wreath would be completed by working the upper and lower faces of the solid to form helical arrises around the cylindrical surfaces to connect the arrises of the straight rails.

Determination of Pitches.

Figs. 278 *and* 281.—In order to determine the inclinations of the upper edges of the tangent planes, the lines of intersection between

the faces of the treads and risers with the vertical tangent planes require to be developed into one plane.

Example 1. *Fig.* 278.—Let the quarter circle *ac* be the plan of the centre line of a wreath turning from a straight flight of stairs on to a level landing. 1 and 2 are the plans of the points in which the lines of intersection between the faces of the treads and the faces of the risers intersect the lower tangent plane. Draw *XY* tangent to the plan centre line at *c*. With *b* as centre, turn 2, *a*, and 1 into the ground line at 2^1, a^2, and 1^1 respectively. Draw $2^1 2^2$ perpendicular and equal to the height of the required riser. Draw the lower pitch through 1^1 and 2^2 to meet *ab* produced in b^1. Draw b^1c^1 horizontal. Then, since the rail is usually made higher over landings and winders than over ordinary full width steps, a^3b^1 and b^1c^1 are suitable pitches. The face mould and bevels are determined in a subsequent figure.

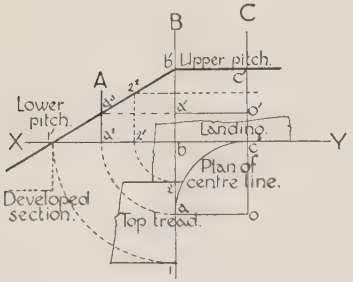


Fig. 278.

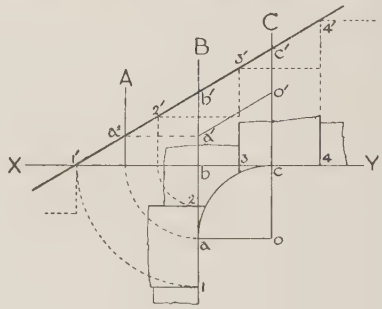


Fig. 279.

Example 2. *Fig.* 279.—The wreath with the best appearance is that in which the pitches are equally inclined, the arrises being true helices. This condition may always be obtained over a quarter space landing if the positions of the first riser in the upper flight and the last riser in the lower flight are not fixed, and it is possible to make the sum of their distances from the line of intersection of the tangent planes equal to the width of the tread. Thus, in the figure the distances 2*b* and *b*3 are together equal to 1-2, or 3-4, so that the developed nosing line $1^1 2^1 3^1 4^1$ is unbroken. The method of development is the same as in the preceding example, and the required pitches are a^2b^1 and b^1c^1.

Example 3. *Fig.* 280.—The quarter circle *ac*, centre *o*, is the plan of the centre line of a wreath over a quarter space of winders. *DE* is the walking line, 18 inches from the plan centre line, and the number of winders is determined by making the distances between the risers

along this line as nearly equal to the full width tread as is possible. The points 1, 2, 3 are turned into the ground line about b, and, starting from any desired one of these points, the riser heights are set up as before to obtain the points 1^1, 2^1, 3^1 . . . 7^1. Draw the springing lines A and C, and the intersection line B. Draw b^1c^1 to pass through 6^1 and 7^1, and a straight line through 1^1 and 2^1. Draw b^1F to pass 2^1 at a distance approximately equal to 5^1 from b^1c^1. Then a^2b^1 is the lower pitch and b^1c^1 the upper pitch.

In this example an easing must be formed between the lower pitch and the lower straight rail; one method of drawing the centre line of the easing is shown and should require no explanation.

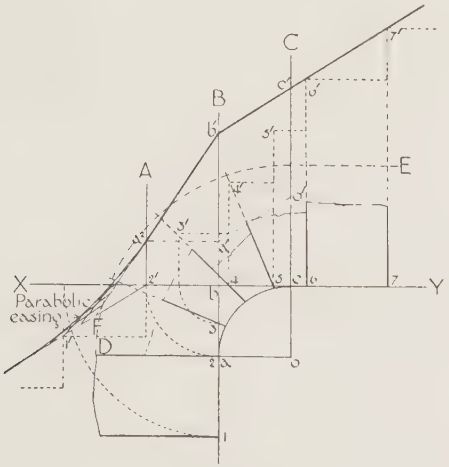


Fig. 280.

Note.—There are no definite rules that can be adhered to in placing the pitches in examples of this type, but since the rail is usually desired higher over the winders than over the fliers, the method shown is considered satisfactory. An easing could also be arranged at the upper end of the wreath, but this is undesirable, owing to it being ungraceful and giving a patchy effect to the completed work. The ideal method is to work the easing in the wreath itself, but this is less usual, owing probably to the difficulty of setting out and the thickness of the material required. An example of the geometrical considerations involved in a wreath worked with an easing at both ends is given later, and if the same method were applied to the present example the falling centre line could be made parallel to the nosing line.

Example 4. *Fig.* 281.—Half turn in two wreaths over a quarter space landing and a quarter space of winders.

The semicircle *ace* is the plan of the centre line, the straight lines *ab, bc, cd, de* are the plans of the vertical tangent planes. The faces of the risers intersect the tangent planes in plan at 1, 2, 3 . . . 8. The tangent planes are developed about *b* and *d* to obtain the nosing line $1^1 2^1 3^1 \ldots 8^1$. Since the distances 1-2 and 7-8 are each equal to a full width tread, the straight lines through $1^1 2^1$ and $7^1 8^1$ are the inclinations of the straight rails. Produce $1^1 2^1$ to cut the springing line *A* in a^2 and meet the intersection line *B* in b^1, and produce $8^1 7^1$ to cut the springing line *E* in e^2 and meet the intersection line *D* in d^1.

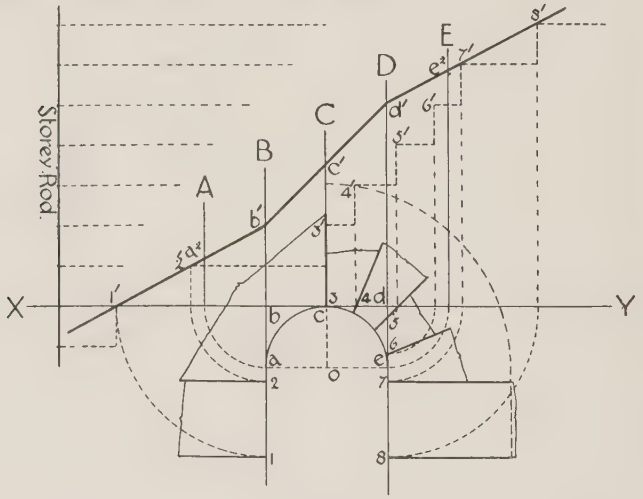


Fig. 281.

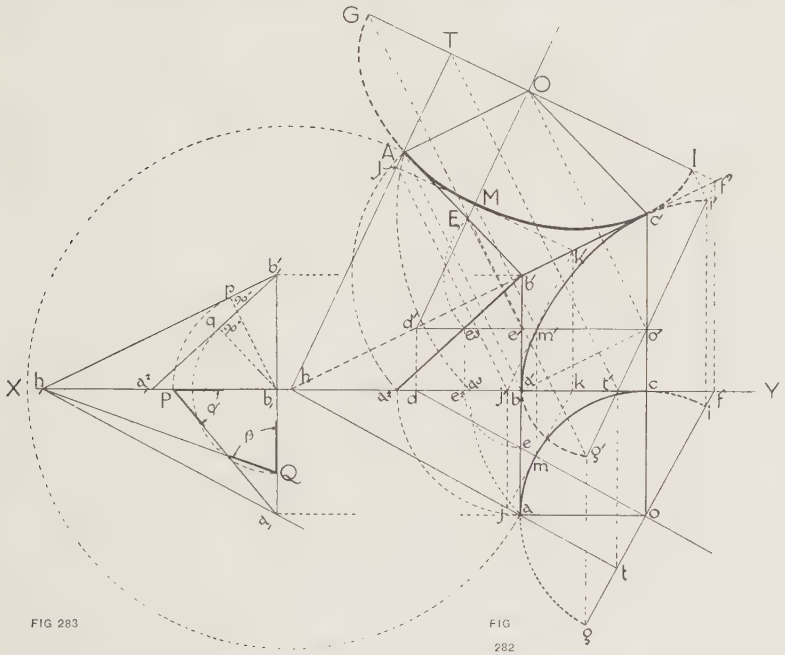
Then the pitches for the lower wreath are a^2b^1 and b^1c^1, and for the upper wreath the pitches are c^1d^1 and d^1e^2. In the following examples the pitches of the upper edges of the tangent planes are assumed to avoid confusion in the diagrams.

Geometrical Principles involved in the General Case.

Figs. 282 *and* 283.—Let the quarter circle *ac*, centre *o*, be the plan of the centre line of a quarter-turn wreath, and let a^1b^1 and b^1c^1 be the elevations of the upper edges of the vertical tangent planes.

Turn *a* about *b* to a^2 and join a^2b^1; then a^2b^1 and b^1c^1 are the pitches. Produce c^1b^1 to meet *XY* in *h*, and draw the straight line *ht* through *a*; then *ht* and hz^1 are the horizontal and vertical traces of the plane

containing the lines AB and BC and the elliptic centre line AC. Draw od parallel to ht, cutting the plan of the centre line in m; then om is the plan of the semi-minor axis of the elliptic centre line. Project d to d^1 and draw d^1o^1 parallel to XY; then d^1o^1 is the elevation of the line do, and m^1 is the elevation of m, and o^1m^1 is the elevation of the semi-minor axis. If e, the point of intersection between ab and od, be turned about b to e^2 in XY, a perpendicular from e^2 will meet o^1d^1 in e^3, its point of intersection with a^2b^1, and e^1, in o^1d^1 and a^1b^1, is the elevation



Figs. 282 and 283.*

of e. Draw jk tangent to the plan circle at m. Project j to j^1 and k to k^1, and draw j^1k^1; then the elevation of the elliptic centre line will be tangent to j^1k^1 at m^1. Draw a perpendicular to ht through o to meet XY in f and the plan circle produced in g and i. Project f to f^1 and draw a straight line through f^1 and o^1 to meet projectors from g and i in g^1 and i^1; then gi and g^1i^1 are the projections of the major axis of the elliptic centre line.

To develop the centre line and tangents into VP. With h as centre, radius ha, draw a circular arc to meet a perpendicular to hc^1 from a^1 in A. Then because ha is the actual distance between the

points h and A, and a^1A is the elevation of the locus of A when turned about h^1c^1 into VP, the straight line b^1A is the development of the lower tangent into VP, and since b^1c^1 is already in VP, Ab^1c^1 is the real angle between the tangents. And since b^1a^2 is also a development of the lower tangent, a circular arc, centre b^1, radius b^1a^2, will pass through A. Let a be turned about c to a^3, in XY. Then, because cc^1 is the height of C above A, the real distance between C and A is equal to the distance between c^1 and a^3; therefore a circular arc, centre c^1, radius c^1a^3, will pass through A. Join a^1o^1; then because a^1o^1 is the elevation of ao and ao is parallel to bc, and since parallel lines project parallel, a^1o^1 is parallel to b^1c^1. Draw hA, and parallel to it draw a straight line from d^1 to meet a perpendicular to hc^1 from o^1 in O. Then o^1O is the elevation of the locus of O when turning about hc^1 into VP, and because do is parallel to ht, d^1O contains the developments of the points whose plans are e, m, and o, and the straight lines joining O to A and c^1 are the developments of the springing lines oa and oc in plan. Since do is the plan of a horizontal line and d^1O is its development, $d^1E = de$, $d^1M = dm$, and $d^1O = do$, when $b^1E = b^1e^3$ and M is in the perpendicular to hc^1 from m^1. Draw perpendiculars to hc^1 from j^1, e^1, and t^1, to cut hA in J, Ab^1 in E, and hA produced in T, respectively; then k^1J is a tangent to the developed centre line at M, and a straight line through T and O will pass through f^1 and be perpendicular to d^1O, and perpendiculars to hc^1 from i^1 and g^1 intersect it in points I and G; and GI is the major axis of the developed centre line.

Note also that $hT = ht$, AO is parallel to d^1c^1, and c^1O is parallel to Ab^1.

The dihedral angles between the oblique plane c^1ht and the vertical tangent planes, required as twist bevels, are shown determined in Fig. 283, as follows:

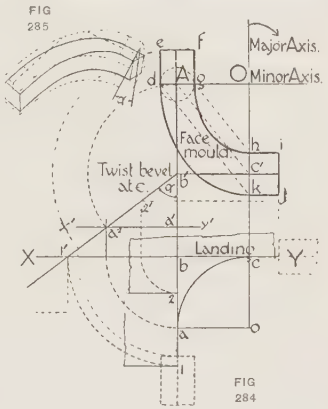
$h_1b^1{}_1$, h_1a_1, and $a^2{}_1b^1{}_1$ are drawn parallel and equal to hb^1, ha, and a^2b^1 (Fig. 282) respectively. With centre b^1, turn p and q to P and Q respectively, b_1p being perpendicular to $h_1b^1{}_1$ and b_1q perpendicular to $a^2{}_1b^1{}_1$, and join Pa_1 and Qh_1 to obtain the angles α and β. Then α is the upper twist bevel and β is the lower twist bevel. A separate figure is introduced to avoid confusion.

The several ways of obtaining the developments of the various points that may be found by considering Fig. 282 are not always alternative in obtaining an accurate result, and each method should be clearly understood by the student of handrailing. Any one method will occasionally fail, either through a point being inaccessible, two lines intersecting at too acute an angle to give a definite point of inter-

section, or for some similar reason; also, each point in a complicated development should be tested for position by a second method of development.

To obtain the Face Mould for a Quarter-turn Wreath with One Inclined and One Level Centre-line Tangent.

Fig. 284.—Let b^1c^1 be the upper tangent and a^2b^1 the developed lower tangent. Because the upper tangent BC is level and the springing line OA is parallel to BC, ao is the horizontal trace of the plane containing the centre line, and the development of the tangent and springing lines is a rectangle having the semi-axes in two of its adjacent sides.



Figs. 284 and 285.

Turn a^2 about b^1 to A in bb^1 produced and draw AO parallel to b^1c^1. With A as centre describe a circle with its diameter equal to the required width of the squared rail. Join Ac^1, and from d and g, the points in which the circle intersects OA, draw parallels to cut Oc in k and h. Then Og is the semi-minor and Oh the semi-major axis of the inside ellipse, Od and Ok the semi-axes of the outside ellipse. The ellipses may be trammelled or drawn by any desired method that gives the true curve.

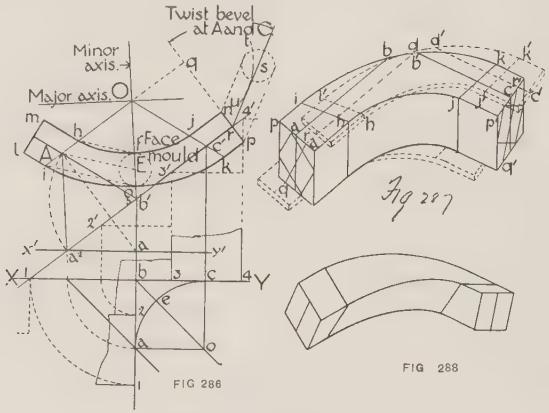
The lengths of the shanks are obtained from the positions of joints in plan, and the joint lines across the mould are drawn perpendicular to the tangents b^1A and b^1c^1. The position of the wreath lying in the plank, and the position for applying the twist bevel, are shown in the sketch, Fig. 285.

It will be seen that the upper twist bevel is the angle made by the straight rail with the vertical, and that the lower twist bevel is a right angle.

To determine the Face Mould and Bevels for a Wreath over a Quarter Space Landing, the Pitches being Equal.

Fig. 286.—The development AOb^1c^1 is obtained in the manner already explained, and, since the pitches are equal, the minor axis is through Ob^1. Make OE equal to the plan radius and draw a circle about E, diameter equal to the width of the rail, to cut Ob^1 in f and g. Join

AE and draw the parallels to meet the springing line in i and h. Corresponding points j and k in the springing line Oc^1 could be determined in a similar manner, but an alternative method which also gives the twist bevel is as follows: Draw the joint line, pn, at right angles to b^1c^1, at any desired position r in b^1c^1 produced, and produce AO and pn to intersect in q. With q as centre, and the plan radius, oa, draw the arc tu. Draw a tangent to this arc from r, and with any point s in the tangent as centre, draw a circle, diameter equal to the width of rail, and draw tangents to this circle, parallel to sr, to meet the joint line in points p and n. Draw nj and pk parallel to rc^1. Then $pnkj$ is the shank of the face mould.



FIGS. 286, 287, and 288.

The angle srq is the twist bevel for both ends in this special example, where the face mould is symmetrical about the minor axis.

In Fig. 287 the face mould is shown in position on each side of the material to give the direction for working the cylindrical edges. The bevels are indicated on the joint planes at prq and $p^1r^1q^1$, the rectangular sections of the rail being shown symmetrically placed about rq and r^1q^1.

Space will not permit a detailed explanation of the method of working the wreath, nor would it be in keeping with the purpose of the book.

Fig. 288 shows the rail squared up ready for moulding.

To determine the Face Mould and Twist Bevels for a Quarter-turn Wreath with Unequally Inclined Pitches.

Fig. 289.—ab, a^1b^1 and bc, b^1c^1 are the projections of the tangents; b^1a^2 is the lower tangent developed into the vertical plane containing

the upper tangent. c^1b^1 produced meets XY in d, therefore da and dc^1 are the traces of the plane containing the tangent lines. oe, parallel to da, is the plan of the line containing the minor axis, and oq is in the plan of the major axis, qoe being a right angle.

Circular arcs, centre d, radius da, and centre b^1, radius b^1a^2, meet in A; then Ab^1 is the lower tangent developed into VP about dc^1. e^1 is the elevation of the point whose plan is e, e^1O is parallel to dA, AO is parallel to c^1b^1, c^1O is parallel to b^1A; then e^1O is equal to eo, and is the development of the line containing the minor axis. AO and c^1O are the developed springing lines. Project q to q^1; then q^1O is per-

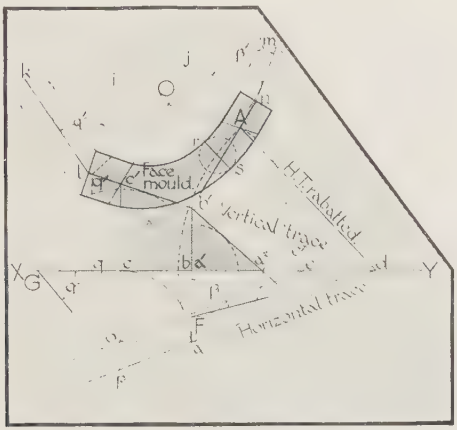


Fig. 289.

pendicular to Oe^1 and is in the major axis. The joint lines at l and n are perpendicular to b^1c^1 and Ab^1 respectively, and are produced to i and j in AO and c^1O produced. kl is tangent to the arc, centre i, radius equal to the plan radius oa, and mn is tangent to the arc, centre j, radius oa; then ilk (a^1) is the angle of twist at the upper end of the wreath, and jnm (β^1) is the angle of twist at the lower end. The face mould is obtained as in the preceding example. If co be produced to meet the horizontal trace in p, and cG be made equal to the perpendicular distance of c from c^1b^1, cGp (a) is the upper twist bevel and is therefore equal to ilk (a^1). If a^1F be made equal to the distance of a^1 from a^2b^1, a^1Fd (β) is the lower twist bevel and is equal to mnj (β^1).

Wreath of more than a Quarter Turn with Unequal Pitches.

Fig. 290.—If the student will compare Figs. 289 and 290 and note the following differences, no further explanation should be required.

The elevation of the lower tangent a^1b^1 does not fall on bb^1. The springing lines *oa*, *oc* in plan, o^1a^1, o^1c^1 in elevation, OA, Oc^1 in development, are not parallel to the tangents. mO and nO are not in the developed springing lines produced, but are parallel to the developed tangents. O and A are obtained by drawing o^1O and a^1A perpendicular to the vertical trace. To obtain the lower twist bevel about *ab*, *bt* is drawn at right angles to *ab*.

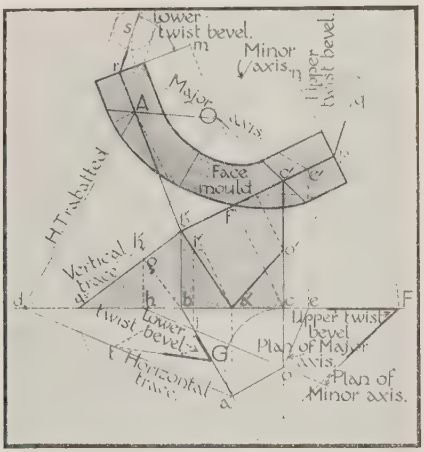


Fig. 290.

Wreath of less than a Quarter Circle with Unequal Pitches.

Fig. 291.—This example differs from the preceding cases in the following particulars:

The horizontal trace passes between *o* and *c*, therefore the centre *o*, the minor axis, the lower springing line *oa*, and the part of the upper springing line between *oa* and *h* are all below *HP*. Then the minor axis, *Ou* in development, is not within the wreath proper, since it does not fall between the springing lines. The widths of the mould on the springing lines may be determined as before, but an alternative method is shown. Thus, with centre *O*, and the plan radius, draw the arc *de*, and draw a tangent to this arc from c^1. Draw a circle, centre *f*, in the tangent, with a diameter equal to the width of the rail. Draw tangents to this circle parallel to fc^1 to meet the springing line in *q* and *r*. Then

q and r are points in the elliptic arcs of the mould. Ou is the semi-minor axis, and points s and t are obtained from the circle about u in the manner explained in some preceding examples.

The upper twist bevel is determined as before, but the lower bevel requires consideration. For, since a perpendicular to ab at b would not intersect the horizontal trace hg within reasonable limits, any other perpendicular, as mp, may be drawn on either side of a, and ml, the perpendicular distance of m from the lower tangent turned to L; then pLm is the required bevel.

Note.—Bb is at right angles to ab and equal to bb^1 and l is in Ba produced.

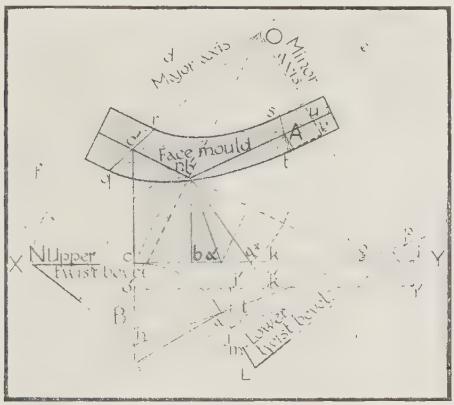


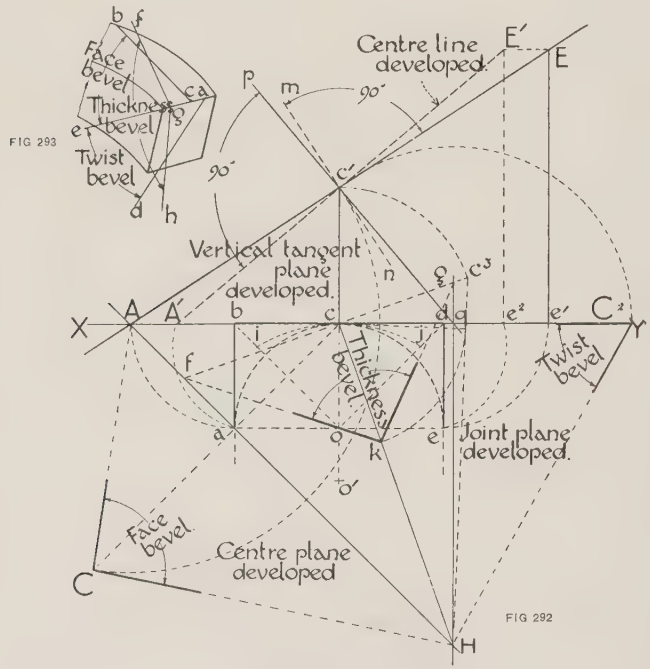
Fig. 291.

The True Normal Joint Plane between Intersecting Wreaths.

Fig. 292.—The joint plane between two wreaths is usually arranged perpendicular to the development of the crown tangents, and in most cases gives a fairly satisfactory result. A joint made in this way, however, is a spliced joint, not a true butt joint, and is not pleasing to the eye of a critical observer.

The only reasons that can be given for adopting the spliced joint are that the true normal joint involves a little more geometrical setting out and requires a little more time in its preparation. In first-class work, however, the slight additional trouble is more than justified by the result. Quite apart from these considerations, the process of finding the three bevels involved in making the normal joint should be studied, since it is the essential thing in the more scientific section of handrailing, namely, forming easings in the wreath.

Refer to Fig. 292. Let the semicircle *ace* be the centre line of a half turn to be made in two wreaths, jointed at *c*. Then *ab* is the plan of the bottom tangent, *bd* is the plan of the common crown tangent, and *de* is the plan of the top tangent. Turn *a* about *b* to *A*, and *e* about *d* to e^1. Draw the vertical line e^1E to represent the difference in the heights of the centre line extremities, and join *AE*. Let *AE* represent the developed edges of the tangent planes. Project *c* to c^1, and draw *mn* through c^1 perpendicular to *AE*; then *mn* is the vertical



Figs. 292 and 293.

trace of the joint plane which is perpendicular to the crown tangent, and since this joint plane is at right angles to the face of the plank, only one bevel, the twist bevel, is required, and this may be found as explained in each preceding example. Now, make oo^1 equal to half *oc*, and with o^1 as centre, o^1c as radius, cut *ob* and *od* in *i* and *j* respectively. With *i* as centre, turn *a* to A^1; and with *j* as centre, turn *e* to e^2; then A^1e^2 is the stretch-out length (approximately) of the semicircle *ace*. Draw e^2E^1 parallel and equal to e^1E, and join A^1E^1. Then A^1E^1 is the development of the helical centre line of the two wreaths. Draw *pq*

perpendicular to A^1E^1 and draw qH perpendicular to XY; then pq, qH are the traces of the joint plane that is normal to the helical centre line. Draw a straight line through A and a to intersect qH in H; then HA and AE are the traces of the plane containing the tangents ab and bc. Join cH.

Now regard the figure AHq as the outline plan of a triangular pyramid, C being the apex and having Ac^1q as its elevation. This pyramid is sometimes referred to as the "handrail pyramid" and is formed by the centre plane of the plank, the plane of the normal joint, and the vertical tangent plane. The vertex C of the pyramid is the point of intersection of the planes, and cH is the plan of the line of intersection between the centre and joint planes.

The three bevels required in making the normal joint are contained in the pyramid as follows: Since the face plane of the plank is parallel to the centre plane, the required twist bevel is the angle between the line of intersection between the centre and joint planes and the line of intersection between the joint and vertical tangent planes; and the development of the joint plane face of the pyramid gives the required angle; thus: with q as centre, turn c^1 to C^2 and join C^2H; then qC^2H is the twist bevel. In the sketch, Fig. 293, ad is the line of intersection between the joint and tangent planes, and ea is in the line of intersection between the joint and face planes; then ead is the twist bevel in position on the material.

Still considering Fig. 293, bc is the line of intersection between the tangent and face planes, and, as already mentioned, ce is the line of intersection between the face and joint planes. Then the angle bce across the face of the plank is found by developing the face plane. Thus, in Fig. 292, cC is perpendicular to AH and AC is equal to Ac^1; then the angle ACH is the bevel across the face of the material. It should be noted that HC equals HC^2.

Referring again to Fig. 293; fg in the face plane is perpendicular to the arris ec, and gh in the joint plane is also perpendicular to ec; then the angle bgh is the dihedral angle between the face and joint planes, and is shown developed in Fig. 292 as follows: Draw fg through c, f and g being in AH and Hq produced, respectively. Produce cg to C^3, making cC^3 equal to cc^1, and join C^3H. With c as centre, draw a circular arc tangent to C^3H and cutting cH in k. Join kf and kg; then the angle fkg is the required thickness bevel.

NOTE.—In the more common joint, the angles bce and fgh, Fig. 293, are right angles.

Method of Forming Easings in the Wreath.

Figs. 294 *to* 298.—In all the preceding examples the centre line of the wreath is wholly in the centre plane of the plank, but in arranging easings in the wreath this simplifying condition cannot be adhered to, since the centre line takes the form of a tortuous curve, such that it can have no straight line projection, and therefore cannot be contained by a plane.

Briefly, the conditions and processes involved are as follows:

Any centre falling line may be arranged and any centre of plank plane may be chosen about the falling line. The required face mould is, as in all the preceding examples, the section of the upright hollow cylinder containing the wreath, by the centre of plank plane. If the necessity for arranging the plane so as to require the minimum thickness of material did not interfere, the problem would be a comparatively simple one.

Consider Fig. 294. 1, 2, 3, 4, 5, 6, 7 are the plans of the positions in which the face planes of the risers intersect the centre plane of the string about a quarter space of winders. The points 1 and 2 are in the straight string leading up to the winders, and 7 is the riser between the winders and the landing.

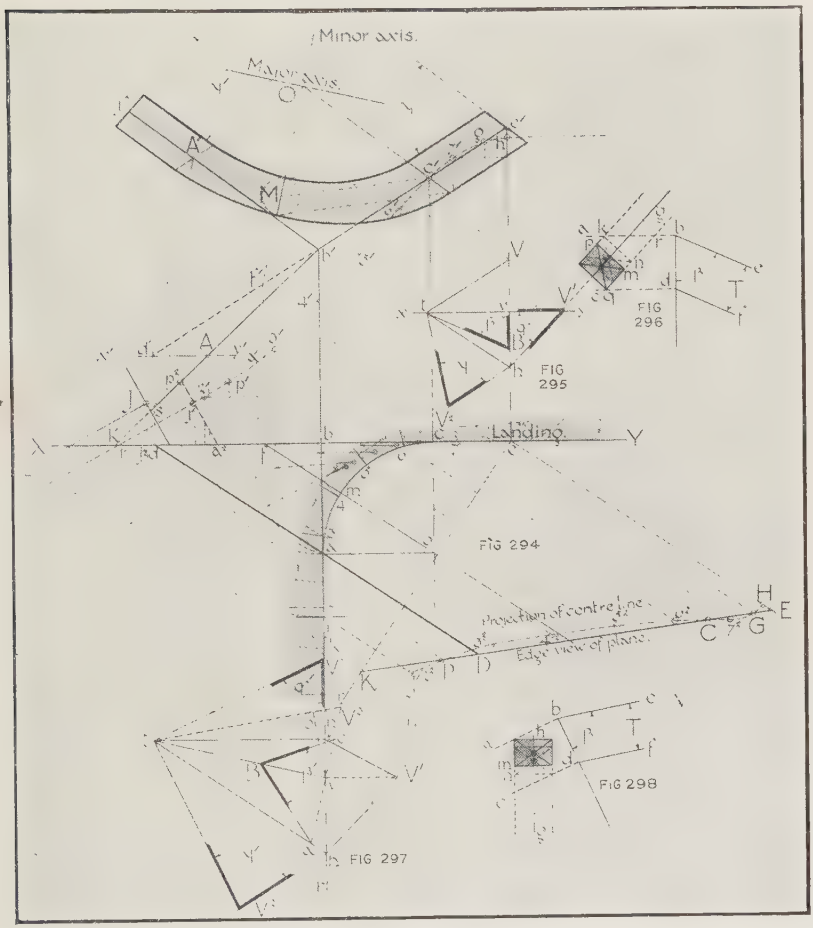
The points 1^1, 2^1 . . . 7^1 are the developments of the points of intersection between the faces of the risers, the faces of the treads, and the vertical centre layer of the string. (In all the preceding examples the development has represented the section of the steps by the tangent planes.) The centres and arcs for developing the points along the plan circle are shown, the method used being that explained in Chapter I.

The dotted line through 1^1, 2^1, 3^1 . . . 7^1 is the development of the selected centre line of the wreath and is made tangential to the upper and lower pitches, the former being level, the latter inclined.

The next step is to select three points in the chosen centre line to be contained by the centre plane of the plank, the three points being selected to give more or less equal deviations of the centre line of the wreath above and below the plane. The qualification "more or less" is imposed by the difficulty involved in selecting the points to give exactly equal deviations, and also because the angles the rail section makes with the face planes of the plank vary from zero at the minor axis to a maximum at one or the other of the extremities of the wreath, and strictly speaking the deviation at the various points should be in inverse ratio of the angles at the points.

No precise rule can be given by which the ideal plane may be determined, since experience and patience are the chief requirements in this operation.

To save time at the expense of material, the extremities of the centre



Figs. 294, 295, 296, 297, and 298.*

line are sometimes selected without further consideration, and in most examples the result is quite satisfactory from every point of view.

In the present example the centre plane has been deliberately arranged at distances from the centre line extremities to afford a better opportunity for explaining the geometric principles involved.

To obtain a satisfactory centre line the joints are arranged at j and e in plan, and at j^1 and e^1 in development, the upper joint plane being vertical, the lower joint plane at right angles to the centre line of the inclined straight rail.

Regard the process of selecting the centre plane, now to be described, as experimental, the result possibly requiring some modification.

Let c^1, the point in which the centre line intersects the perpendicular from c, be selected as a point in a trial plane, and draw the vertical trace of the plane to cut the centre line in a second chosen point, g. Turn j about b to j^2 and draw a perpendicular to XY from j^2 to meet a horizontal from j^1 in J. From J draw the lower pitch parallel to the nosing line of the straight flight, and perpendicular to it draw the joint line. From b^1, in the perpendicular from b and in gc^1 produced, draw b^1k^1 to meet the joint line in any chosen point s. Turn a about b to a^2 and draw a perpendicular from a^2 to cut b^1k^1 in A. Draw x^1y^1 through A to meet the vertical trace in d^1. Project d^1 to d and draw the horizontal trace da. Draw eK perpendicular to da, and regarding eK as a ground line, project an elevation of the centre line 1^2, 2^2 . . . H, and draw the edge view of the trial plane KE. The distances of the points in this view from eK are equal to the heights of the corresponding points above XY. Since KDE is an edge view of the plane, the elevation of the centre line shows the positions and magnitudes of its deviations from the centre plane ; and it will be seen that the thickness of the required plank might be reduced by rearranging the plane so as to bring it nearer to the extremities J^1 and H, since the greater twist at these positions brings the arrises of the rail nearer to the faces of the plank.

To effect any rearrangement for bringing both extremities of the centre line nearer the centre plane, the simplest method would be to turn the plane downwards about the horizontal trace so as to bring the upper extremity of the centre line into the centre plane, since this would not entail a new projection of the centre line and would bring the lower extremity J^1 also nearer to the centre plane. To obtain the most economical plane, however, the vertical trace would require lowering to a position nearly parallel to its present position, but cutting the centre line development in a point between g and h, and in a second point between $c^1 6^1$. The student should now be able to determine this plane by the trial process explained ; but the plane already determined in the figure is very satisfactory for the purpose of explanation.

The face mould curves and tangent development are obtained as in the preceding examples.

The Joint Bevels.

Three bevels are required for each joint, namely, a twist bevel, a face bevel, and a thickness bevel; and these are determined in the same manner as for the normal joint between two wreaths.

Consider Fig. 295. vth is the plan of a triangular pyramid, v being the plan of the vertex. The base of the pyramid is in HP, the three remaining faces being in, or parallel to, the vertical joint plane, the vertical tangent plane, and the centre plane of the plank. xy is any line perpendicular to ee^1; th is any line parallel to da (Fig. 294); intersecting xy in t, and ee^1 in h. Draw $t\,V$ parallel to d^1e^1 (Fig. 294), and with v as centre, turn V to V^1 in xy. Join V^1h, and with t and h as centres, $t\,V$ and $h\,V^1$ as radii respectively, draw circular arcs to intersect in V^2. Join V^2, h and V^2, t. Then $v\,V^1h$ is the developed joint plane face of the pyramid and the angle $v\,V^1h$ (α) is the required twist bevel. $t\,V^2h$ is the developed centre plane face of the pyramid and the angle $t\,V^2h$ (γ) is the required face bevel. With v as centre draw a circular arc tangent to $h\,V^1$, to cut vh in B, and join Bt. Then vBt (β) is the dihedral angle between the centre and joint planes and is the required thickness bevel. The joint line across the face mould through e^1 is drawn to make the angle γ with the tangent line b^1e^1, but since the joint plane is vertical the joint line is parallel to the springing line Oc^1.

Bevels for the Lower Joint.

Fig. 297.—From any point h in ba (Fig. 294) produced, draw ht parallel to da (Fig. 294) and any convenient length, and draw th^1 parallel to XY. Draw ru to make an angle with hh^1 equal to the pitch of the lower straight rail, and draw $h^1\,V^1$ at right angles to ru. Draw $h\,V^1$ to make an angle with hh^1 equal to the inclination of the lower tangent (bk^1b^1, Fig. 294). Draw V^1v perpendicular to hh^1, and join vt. Then hth^1 is the plan of a triangular pyramid, vertex v, having its base in HP, one face in the vertical plane of the lower tangent, and the two remaining faces parallel to the centre plane and the joint plane respectively.

With h^1 as centre, turn V^1 to V, and with t and h as centres, $t\,V$ and $h\,V^1$ as radii respectively, draw circular arcs to intersect in V^2. Join V^2t and V^2h. Draw ac through v perpendicular to vt to meet th in a, and th^1 produced in c. With v as centre, $v\,V^1$ as radius, cut ac produced in V^3 and join V^3t. Draw vb at right angles to V^3t, and with v as centre, vb as radius, cut vt in B. Join Bc and Ba. Then $t\,Vh^1$ is the developed joint plane and a^1 is the required twist bevel. $t\,V^2h$ is the

developed centre plane and γ^1 is the required face bevel. β^1 is the dihedral angle between the centre and joint planes, and is the required thickness bevel.

Draw the joint line across the face mould (Fig. 294) at J^2 to make the angle γ^1 with the tangent J^2A^1.

Thickness of Plank.

Fig. 298.—Produce $h\,V^1$ (Fig. 295) to any point g, and with g as centre draw a circle of diameter equal to the width of the rail. Draw ab horizontal, to cut tangents to the width circle, parallel to hg, in k and r. Draw kn at right angles to gh. Make the distances kp and nm each equal to twice the centre displacement e^1h (Fig. 294), and make mq equal to the rail thickness. Draw cd through q, bd being perpendicular to ab. Draw be to make the angle β with bd, and draw df parallel to be. Then the perpendicular distance T is the required thickness of plank.

In Fig. 298 the process is repeated about the lower twist and thickness bevels and with twice the centre line displacement at the lower end of the wreath, but since this gives a lesser thickness than at the upper end, it would only serve where all the arrises are not required in the squared rail section.

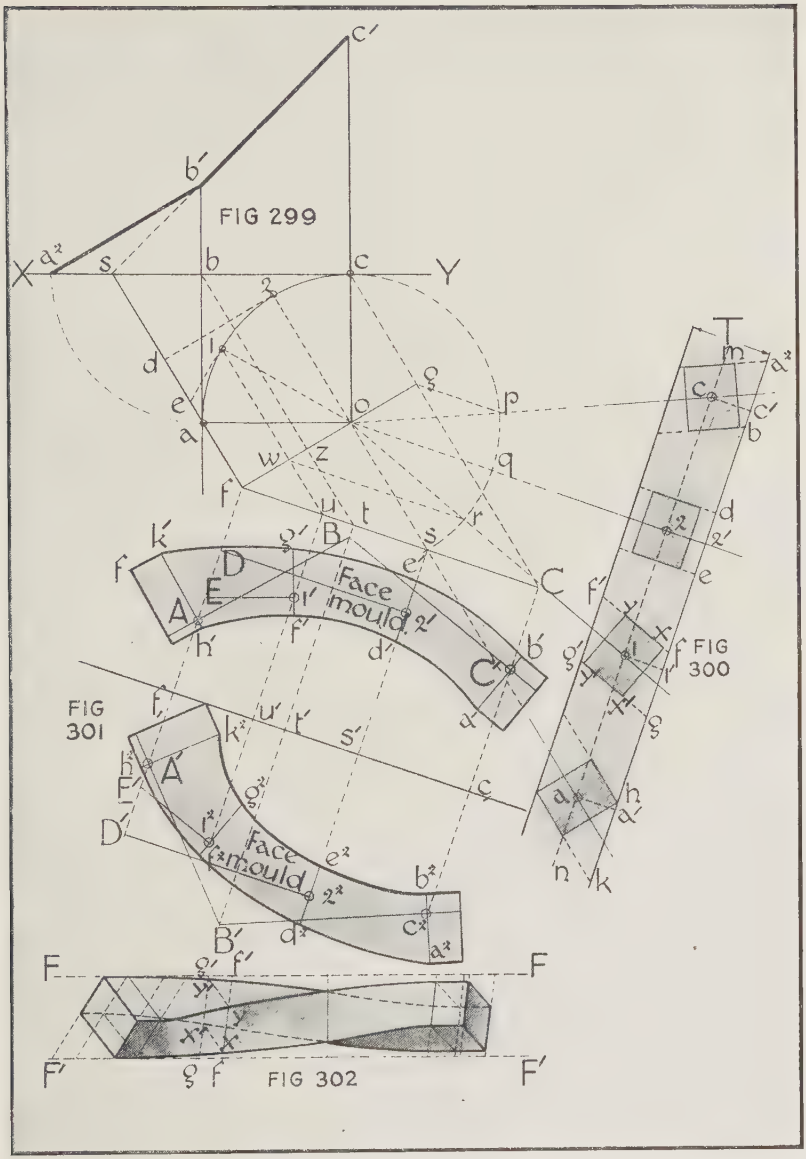
Falling moulds, only required in this type of wreath, are simply the developments of the inner and outer cylindrical surfaces of the wreath, and may be obtained to conform to the centre falling line by methods already explained.

Handrailing by the Method of Normal Sections.

Figs. 299 *to* 302.—The method consists of finding the twist bevels at any desired number of points along the wreath, plotting the positions of normal sections at these points relative to the faces of the plank, and obtaining two moulds by plotting the positions of the points in which the section edges meet the faces of the plank relative to the projections of the centres of the sections on to the faces of the plank.

Let the quadrant, ac, be the plan of the centre line of a quarter-turn wreath, a^2b^1, b^1c^1 the inclinations of the upper edges of the tangent planes. Draw fg perpendicular to the horizontal trace sa, and draw cgC parallel to sa, making gC equal to cc^1, and join fC. Then fC is an edge view of the plane containing the centre line of the wreath.

Select any points, 1 and 2, in the plan of the centre line, and draw tangents to meet Sa in d and e. Draw parallels to sa from 1, b, and 2, to meet fC in u, t, and s respectively Let w be the point in which



Figs. 299, 300, 301, and 302.*

$1u$ cuts fg, and draw wr, oq, and gp parallel to fC; p, q, r, being in the plan circle produced. Let s be the point in which the plan circle cuts fC. Join op, oq, or, and os.

Draw mn, Fig 300, perpendicular to fC at any convenient distance from Fig. 299, and produce op, oq, or, and os in the latter figure to cut mn in points c, 2, 1, and a. Then the angles between mn and these radiating lines are the angles of twist at the four points shown in plan at a, 1, 2, and c, the letters and numbers corresponding in the two figures. About a, 1, 2, and c (Fig. 300) as centres, set out the rail sections with their edges parallel and perpendicular respectively to the radial lines. Draw parallels to mn through the two angular points which are the most distant from mn to represent the faces of the plank, and produce the thickness edges of each section to meet either face in a^2, b, d, e, f, g, h, and k.

Project c, 2, 1, and a perpendicularly on to the face at c^1, 2^1, 1^1, and a^1 respectively.

Draw fc, Fig. 301, parallel to and at any distance from fC, Fig. 299, and draw perpendiculars to fC from f, u, t, s, and C, cutting f_1c_1 in f_1, u_1, t_1, s_1, and c_1 respectively. Make f_1A and f_1A^1 each equal to fa (Fig. 299); f_1E and f_1E^1 each equal to fe (Fig. 299); f_1D and f_1D^1 each equal to fd (Fig. 299); and so for $u^11^1 = u^12 = w1$ (Fig. 299); $t^1B = t^1B^1 = zb$ (Fig. 299); $s^12^1 = s^12^2 = o2$ (Fig. 299); and $c_1C^1 = c_1c^2 = gc$ (Fig. 299). Draw the developed tangents AB, A^1B^1, $E1^1$, E^11^2, $D2^1$, D^12^2, and BC^1, B^1C^2. Draw k^1h^1 through A perpendicular to BA, g^1f^1 through 1^1 perpendicular to $E1^1$, d^1e^1 through 2^1 perpendicular to $D2^1$, and a^1b^1 through C^1 perpendicular to BC^1. Draw corresponding perpendiculars through A^1, 1^2, 2^2, and c^2. Make Ak^1 and A^1k^2 each equal to a^1k, Fig. 300; Ah^1, A^1h^2 each equal to a^1h, Fig. 300. Proceed in the same way to obtain g^1g^2, f^1f^2, e^1e^2, d^1d^2, a^1a^2, and b^1b^2, each being obtained from Fig. 300, as indicated by corresponding letters and figures. Draw curves through $k^1g^1e^1b^1$, $h^1f^1d^1a^1$, $k^2g^2e^2b^2$, and $h^2f^2d^2a^2$. Add any desired length of shank to the ends of the moulds and draw the joint lines perpendicular to the tangents from B and B^1. Since the distances transferred from Fig. 300 to obtain the face mould curves are set off on the opposite sides of the corresponding tangents, the two moulds are of different shape, and when placed on opposite sides of the plank with corresponding tangents directly opposite each other the edges of the rail are obtained without sliding the moulds.

After working the material to the moulds, the arrises may be accurately determined by setting off along the lines joining the extremities of corresponding section lines on the moulds the distances of the angular points from the faces of the plank, these distances being taken

PLATE X.

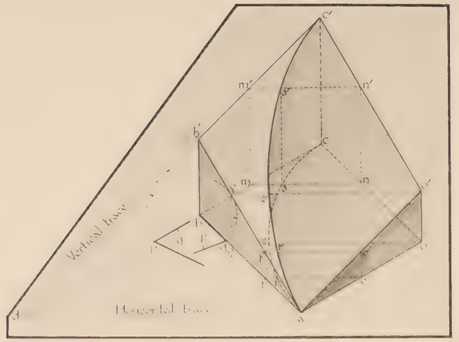
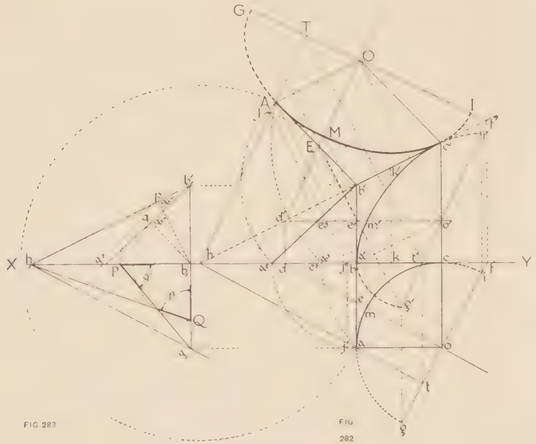
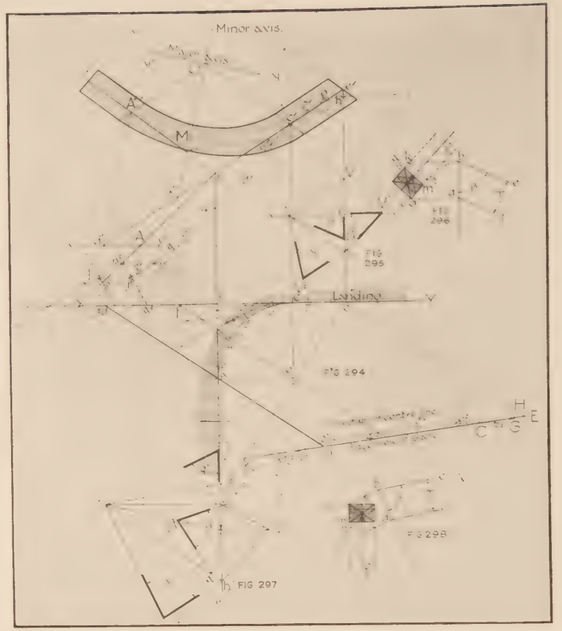


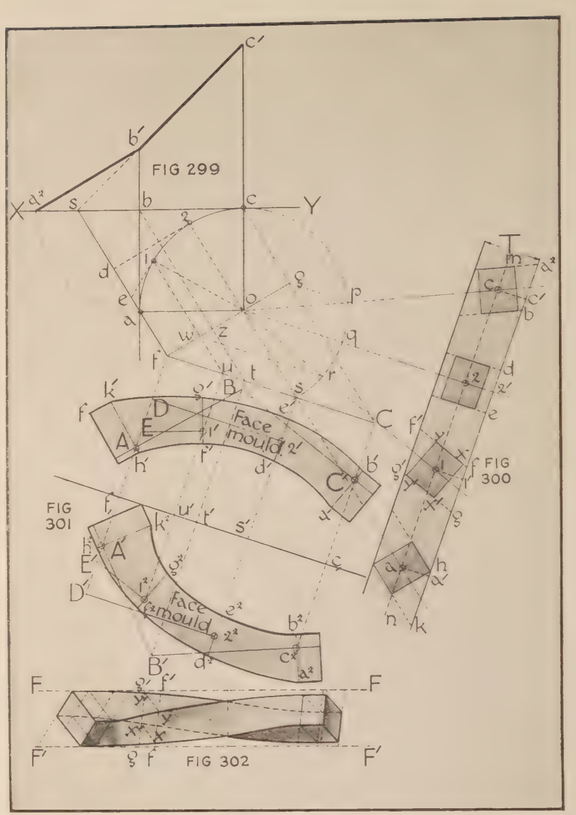
FIG. 276.*



FIGS. 282 and 283.*



FIGS. 294, 295, 296, 297, and 298.*



FIGS. 299, 300, 301, and 302.*

face page 276.

from Fig. 300. Thus, let FF and F^1F^1, Fig. 302, be the faces of the plank, xx^1yy^1 be a view of the section 1, then fx, f^1y^1, gx^1, g^1y (Fig. 302) equal fx, f^1y^1, gx^1, and g^1y (Fig. 300) respectively.

Fig. 302 shows the positions of the four sections in the squared wreath relative to the plank faces, the view being taken parallel to section 2 on the minor axis, where no twist occurs.

CHAPTER XVII

PLACING LINES DIRECT ON TO THE MATERIAL

Figs. 303 *to* 311.—The absence of facilities for setting out elaborate geometrical constructions on the building site is the chief factor in perpetuating crude rule-of-thumb methods, and is responsible for much of the poor craftsmanship to be found in quite important structures. The advantages of placing the lines accurately on the material by direct measurement should not require emphasising.

The examples selected are ordinary roof members, but the principle involved has a much wider application.

Fig. 303 is the plan of one corner of a hipped roof, the sizes of the members being made large in proportion to their lengths to show the angles clearly.

Fig. 304 is a section through the roof showing the inclination. The angle LCR is the lower bevel of the common rafter, UCR the upper bevel.

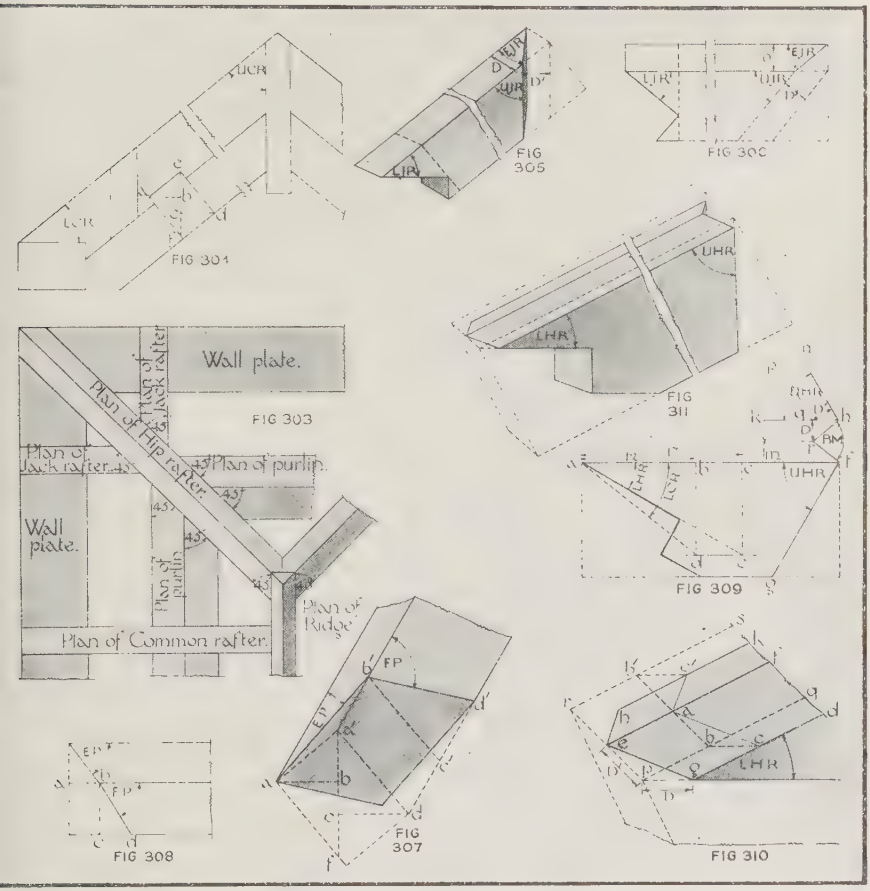
Given either of these bevels, the bevels for the purlins, jack rafters, and the hip rafter may be placed directly on the material without the aid of the steel square, and without the necessity of memorising any formulæ.

First consider the jack rafter, Fig. 305. The lower bevel LJR is the same as LCR, and the upper face bevel UJR is the same as UCR. The upper edge bevel EJR is 45° *in plan*; therefore, if the horizontal distance D^1 is made equal to the thickness D, the bevel EJR is obtained. This is shown more clearly in Fig. 306, in which the face and edge are brought into one plane.

Next consider the purlin, Fig. 307. Referring again to Fig. 304, it will be seen that the straight line *ef*, drawn vertically down the end section of the purlin, makes with the arris *ae* an angle equal to UCR. Let *ab* and *cd* be horizontal lines from the angular points *a* and *d* of the section, meeting *ef* in *b* and *c* respectively. These horizontals are shown in Fig. 307 at *ab* and *cd*, and since the face and edge bevels of the purlin are each 45° *in plan*, the bevels are obtained on the material

by making the distance a^1b^1 equal to ab, squaring b^1 to c^1, and making c^1d^1 equal to cd. Fig. 308 shows a face and edge brought into the plane of the paper, the distances ab and cd being shown more clearly in position.

Now consider the face bevels of the hip rafter, Fig. 309.



Figs. 303, 304, 305, 306, 307, 308, 309, 310, and 311.*

Because the diagonal of a square of 12 in. side is equal to 17 ins., or very nearly, since $12^2+12^2=288$, and $17^2=289$, the rise of the hip rafter in 17 ins. horizontal run is equal to the rise of the common rafter in 12 ins. horizontal run; since the angle between the plan of common rafter and the plan of the hip rafter is 45°.

From a draw ad to make the angle LCR with the upper edge of

the hip rafter. Make *ab* and *ac* equal to 12 and 17 units respectively. Draw *bd* and *ce* square to the edge of the hip rafter, and make *ce* equal to *bd*; then *ae* makes the required angle *LHR* with *ac*. The upper face bevel *UHR* is determined by making *fg* perpendicular to *ae*.

The angle required for the birdsmouth joint between the hip and the ridge is determined in the same figure at *BM*. Let *fmkh* be the edge of the hip developed into the plane of the face, and let *nhk* be part of the second face developed. Then if *pq* be drawn parallel to *nh* at a distance D^1 equal to *D*, half the thickness of the hip, cutting the upper arris in *q*, a square line across the edge from *q* to meet the centre line in *r* determines the required angle *BM*, since each of the angles *rhk*, *rfm* appear at 45° *in plan*. No developments are required in the actual process of course; they are only used for the purpose of explanation.

The backing bevel for the hip rafter is the dihedral angle between the intersecting roof surfaces, and may be determined by direct measurement as shown in Fig. 310. The dotted lines *pq* and *rs* represent the inclined arrises of the upper edge of the hip rafter before the process of backing is carried out, and bb^1 is any line across the edge square to these arrises. *ef* is the centre line of the upper edge, and *bc* and b^1c^1 are horizontal lines on the faces of the hip. Since *bc* and b^1c^1 are each parallel to *pq* they may be drawn with the bevel set to the angle *LHR*, Fig. 309. If *bc* and b^1c^1 are each made equal to *ab*, half the thickness of the hip, lines *gd*, *hk*, through *c* and c^1 parallel to the arrises *pq*, *rs*, give the backing angle. For, the angles *gef* and *hef* each appear at 45° *in plan*. therefore the distance *D* equals the distance D^1, and since *pg* is parallel to *bc* and *ep* is parallel to *ab*, any horizontal section, as cac^1, through the backing angle must be equal to the angle *heg*.

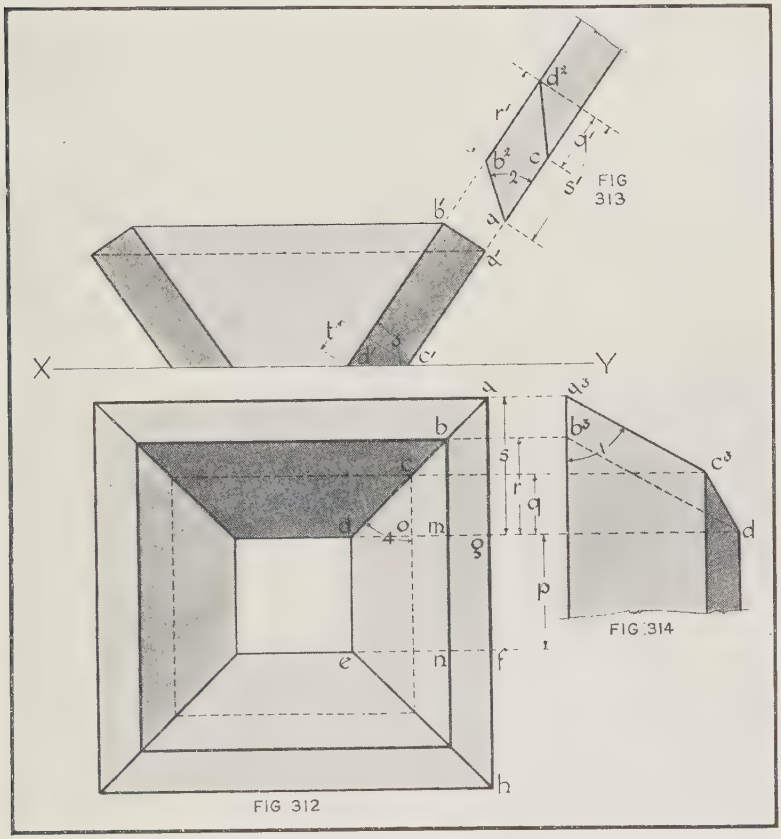
The completed hip rafter is shown in pictorial view at Fig. 311. The 90° angle for the abutment of the hip on to the inner vertical surfaces of the wall plates is not shown, being of too simple a nature to require explanation.

A second method of obtaining angles by direct measurement on to the material is shown in Figs. 312 to 314. The method consists of setting out arris lengths from chosen lines square across the material, and may be applied to many types of construction.

The example selected for the purpose of explanation is a square hopper with inclined sides and mitred angles, the plan and elevation being given as shown in Fig. 312. The angles required are a face bevel and two edge bevels. It will be seen that the upper edges are at right angles to the faces, and the lower edges are in one plane.

To obtain the face and edge bevels, 1 and 2, respectively, for cutting the mitre.

Prepare the material to the exact width b^1d^1. Set out the distance p equal to the length de on the intended lower inner arris, from d and e. Square lines across the inner face to m and n, and across the upper



Figs 312, 313, and 314.

edge to f and g. Along the inner upper arris set outwards from m and n the distance r, to obtain the point b. On the upper outer arris set out the distance s from g and f, to obtain the points a and h. Join bd and ba. A line from a across the outer face parallel to bd completes the lines for cutting the mitre. The angles may be transferred to the other ends with the bevel.

To bevel the lower edge, set the marking gauge to t and gauge from

the lower edge along the outer face. If the material is already bevelled, as would probably be the case in a machine shop, the angle across that edge would be exactly as seen in plan at *ocd* (4).

Fig. 313 shows the mitred end resting on its square edge, and the arris projections s^1, q^1, r^1 are equal to s, q, r respectively. The true edge bevel is marked 2.

Fig. 314 shows the mitred end resting on its inner face, the angle 1 being the true face bevel.

CHAPTER XVIII

MOULDINGS: SECTIONS, INTERSECTIONS, AND PROJECTIONS

THE necessity for obtaining intersections between level and raking mouldings on vertical mitre planes results in one or both of the moulding sections being distorted, but since the actual sections of the mouldings are rarely to be seen when the mouldings are mitred and fixed, the distortion does not necessarily mean an unsightly arrangement. The vertical mitre plane is necessary in all cases where mouldings are carried around vertical wall surfaces.

Section of a Raking Moulding to intersect a Level Moulding of given Section on a Vertical Mitre Plane, the Angle between the Plans of the Mouldings being given.

Fig. 315.—AB . . . MN is the given section of the level moulding. The plans of the two mouldings are shown intersecting on a vertical mitre plane, a^2n^2, which bisects the angle between the plans.

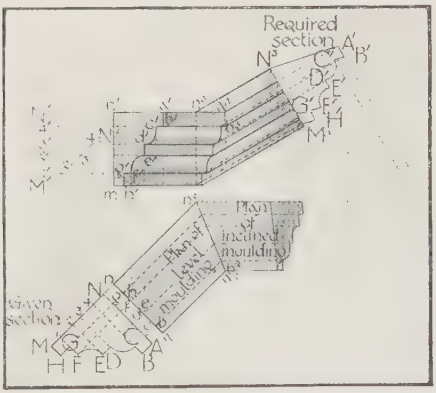


FIG. 315.

Make M^1N^1 equal to MN and determine the points 1^1, 2^1, 3^1, 4^1 to represent the projections of points of intersection between the members on to the back of the moulding; these points may be obtained from MN as shown.

Draw the plans of the lines of intersection between the members meeting the vertical end section an in c, d, e . . . h. Draw perpendiculars from a . . . n to meet horizontals from M^1, 1^1, 2^1 . . . N^1 in a^1, b^1 . . . n^1, and thus obtain the elevation of the vertical end

section. Draw the elevations of the horizontal arrises of the moulding lying between the end and mitre planes, as a^1a^3 and b^1b^3. From the points thus obtained in the mitre plane draw the inclined arrises at the required inclination. Draw M^3N^3 perpendicular to the inclined arrises and set out along the arrises, produced beyond M^3N^3, distances equal to the projections of the arrises in front of the plane containing the back of the moulding; these distances may be determined from the plan as shown.

The required section is shown at $N^3A^1B^1M^3$, and it should be noted that parallel members in the given moulding result in corresponding parallel members in the required section; thus N^3A^1, C^1B^1, F^1G^1, and M^3H^1 are all parallel, as are also N^3M^3, G^1H^1, D^1E^1, and A^1B^1. The plan of the right-angled section M^3N^3 of the required moulding is shown. Only the elevation of the mitre section is required in determining the required section, but the additional sections are projected here to obtain a more helpful diagram.

Projection of Two Intersecting Inclined Mouldings.

Fig. 316.—The plan and elevation of two mouldings, each inclined at 30°, intersecting on a vertical mitre plane, are shown in the figure.

Let the plan angle cc^3q, and the section $ABKM$, of the lower moulding be given.

Draw A^2a^3 to make an angle with Aa^3 equal to the inclination of the moulding, and draw AM^2 perpendicular to A^2a^3. Project the angular points and any required points in the curved members on to MA, and turn about A into AM^2. Draw perpendiculars to Aa^3 from the points in AM^2 to meet the plan arrises in m, k, g, f, e, d, and c; then, because the moulding is inclined at the angle Aa^3A^2 and M^2A is perpendicular to A^2a^3, M^2A may be regarded as the rabatment of the back edge of the section about a horizontal line through A into the plane of the paper, and therefore $ABcde \ldots km$ is the plan of a right-angled section of the inclined moulding. Also A^2a^3 may be regarded as the rabatment of the upper back arris of the moulding about a horizontal line through a^3, and therefore the distance X along BA produced is the difference in the heights of A and a^3. Project a^3 to any point a^2, and project A to a^1, making the difference X^1 in the heights of a^1, a^2 equal to X; then a^1a^2 is the elevation of the upper back arris of the moulding.

Project m to m^1, making the difference Y^1 in the heights of a^1, m^1 equal to Y, then a^1m^1 is the elevation of the arris AM; for, since Am,

a^1m^1 are the projections of a line whose real length is AM^2, Y is the difference in the heights of its extremities.

Project B to b^1, making a^1b^1 horizontal. The elevation of the complete section may now be obtained by direct projection, as shown. The problem is now the same as in the preceding example, the chief point to remember being that parallel lines project parallel.

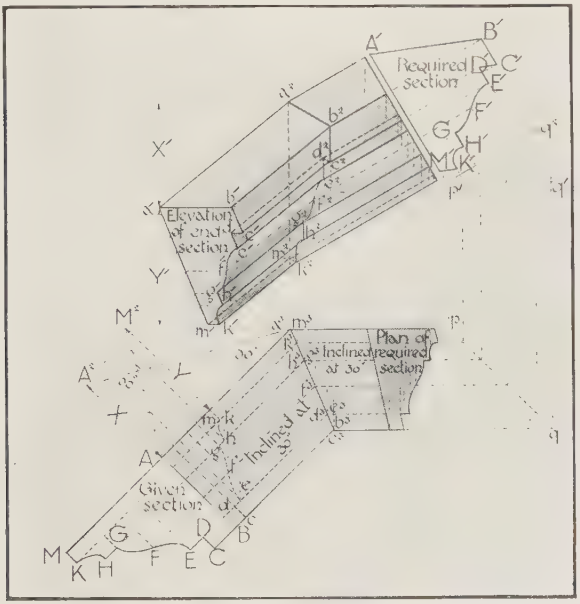


Fig. 316.

Development of Moulding Machine Cutters.

Fig. 317.—Scraper cutters used with the slotted head on vertical spindle moulding machines require no development, since the face of the cutter is almost perpendicular to the face of the fence at the position of maximum projection. In working with the square block on the vertical spindle, and on multiple cutter moulding machines, however, the face of the cutter is inclined to the face of the fence at the position of maximum projection, and therefore the working edge of the cutter is not an exact reverse of the moulding it produces, and for any given section the shape of the cutter must be determined.

Consider Fig. 317. The plan of the cutter and spindle head are shown at the position in which the face of the cutter is at right angles to the face of the fence. e^2 is the plan of the most forward point E

in the edge of the cutter, and the circular arc through e^2, centre O, is the plan of the path of E when the spindle is revolving. The plan of point E at the position of maximum projection is e, where Oe is perpendicular to the face of the fence. Since all points in the cutting edge have circular paths similar to that of E, each point will be at the position of maximum projection when its plan is in the line Oe. Then the cutter required to produce the moulding shown in section at *abcde* is determined by projecting the angular points, and any additional points in the section, on to Oe, turning these points about O into the cutter face, and making the vertical distances down the cutter face equal the vertical distances of corresponding points in the moulding section. Thus, a^1, b^1, c^1, d^1, 1^1, 2^1, e are the projections of the points a, b, etc., in the section on to Oe, and a^2, b^2 . . . e^2 are the positions of the points when turned about O into the cutter face. A, B, C, D, 1^3, 2^3, E are in perpendiculars to a^2e^2, from the points a^2b^2 . . . e^2, and the vertical distances of A, B . . . E from FE are equal to the vertical distances of the corresponding points in the moulding section from a^1e; thus the distance of D below FE is equal to d^1d, and similarly for the remaining points. Then $ABCDE$ is the shape of the required cutter.

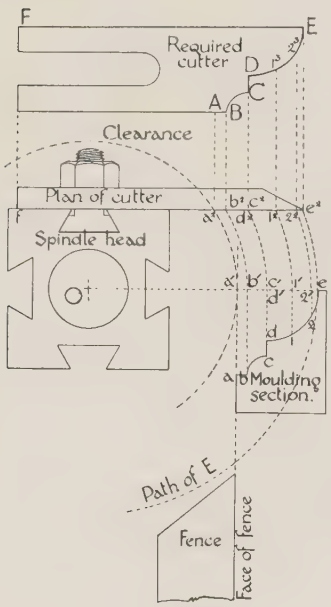


FIG. 317.

It will be seen that the outline of the cutting edge is equal to the section line between a vertical plane containing the face of the cutter at any position throughout its rotation, and a surface of revolution generated by the section line of the required moulding, lying in a vertical plane through O, and at the chosen distance from O, revolving about the vertical axis through O.

The position of the fence is determined by the projection of the bolt holding the cutter, a minimum safe clearance being shown in the figure.

CHAPTER XIX

PROJECTION OF SHADOWS

The projections of shadows cast by opaque bodies may be determined, generally, by finding the traces of straight lines, the lines representing light rays, on the surface or surfaces upon which the shadow falls, and drawing the shadow outline through these traces. This is true for either parallel or diverging light rays.

For the purpose of shadow projection the sun's rays are assumed to be parallel, and a single artificial light is usually considered as the point of divergence of non-parallel rays.

In architectural work the projections of the light rays are generally assumed to be at 45° in both plan and elevation, and this somewhat simplifies the process of shadow determination.

Shadow cast by a Point.

Fig. 318.—Let pp^1 be the projections of the point r, and r^1 the projections of the direction of the light rays.

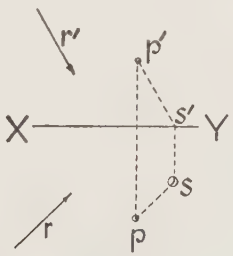


Fig. 318.

Draw p^1s^1 parallel to r^1 and ps parallel to r. Project s^1 to s; then s is the horizontal trace of the line through P parallel to the light rays, and is the required shadow.

Shadow cast by a Straight Line on to the Co-ordinate Planes.

Fig. 319.—ab, a^1b^1 are the projections of the line. Determine d and e, the horizontal traces of the rays through A and B respectively, and c^1, the vertical trace of the ray through B. If the vertical plane did not intervene, de would be the shadow of AB on HP, therefore df is the length and position of the shadow on HP and fc^1 is the shadow on VP. If a straight line be drawn from f parallel to r, it will cut ab in the plan of the point whose shadow falls on XY.

Shadow of a Straight Line on an Oblique Plane and the Co-ordinate Planes.

Fig. 320.—The projections of the line are ab, a^1b^1. vth is the given plane. Determine p, the horizontal trace of the ray through A; c, the horizontal trace of the ray through B; and f^1, the vertical trace

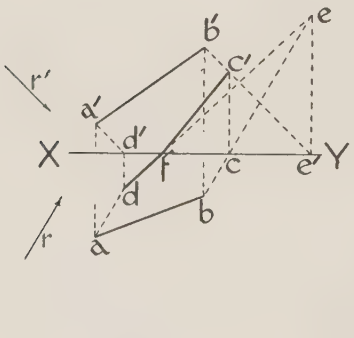


Fig. 319.

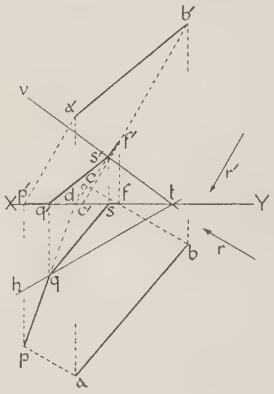


Fig. 320.

of the ray through B. Draw pc to cut ht in q and xy in d, and draw df^1 to intersect vt in s^1. Project s^1 to s and q to q^1. Then pq is the shadow on HP, f^1s^1 is the shadow on VP, and qs, q^1s^1 are the projections of the shadow across the oblique plane, vth.

The Light Area on Two Vertical Wall Surfaces and a Horizontal Floor Surface by Parallel Rays through a Semicircular Headed Opening in a Third Vertical Wall Surface.

Fig. 321.—$abde$ is the plan, $a^1b^1d^1e^1$ the elevation of the opening; pq is the plan of a vertical surface parallel to the plane of the opening, and qm is the plan of a vertical surface perpendicular to the plane of the opening. o, o^1 are the projections of the centre of the semicircle.

Determine c^1, the vertical trace of the light ray through O, and s^1 the vertical trace of the ray through D. With c^1 as centre, c^1s^1 as radius, draw the circular arc s^1q^1. Draw s^1n^1 parallel to a^1d^1. Then $n^1s^1q^1$ is the line of separation between light and shade on the wall surface pq, and because the plane of the opening is parallel to the plane of the wall surface, s^1c^1 is equal to o^1d^1. From any points i and j in mq draw parallels to r to meet ab in g and h, and project g and h to g^1 and h^1 respectively. Draw ef parallel to r, and turn i, j, and f to i^1, j^1, f^1 in pq produced. Draw g^1i^2, h^1j^2, and e^1e^2 parallel to r^1, and draw perpendiculars from i^1, j^1, and f^1 to meet horizontals from i^2, j^2, and e^2 in I, J, and F respectively. Draw the elliptic curve through q^1, I, J, F

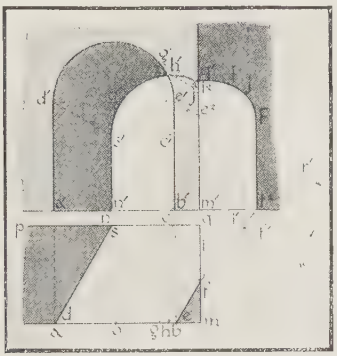


Fig. 321.

and the perpendicular Ff^2. Then $qIJFf^2$ is the line of separation between light and shade on the wall surface, MQ, rabatted into the plane of PQ.

Shadow cast by a Straight Line on to a Sphere.

Fig. 322.—ab, a^1b^1 are the projections of the line; o, o^1 the projections of the centre of the sphere.

Determine p and s, the horizontal traces of the rays through A and B respectively, and join ps. Then ps is the horizontal trace of the plane containing AB, Bs, and Ap, and part of the line of intersection between this plane and the surface of the sphere is the required shadow. Draw x^1y^1 perpendicular to ps to touch the plan outline of the sphere, and draw aa^2 parallel to ps, making the height of a^2 above x^1y^1 equal to the height of a^1 above XY. Join x^1a^2. Then x^1a^2 is an edge view of the plane and vu is equal to the diameter of the section of the sphere by the plane. Project o on to the plane at o^2, and turn o^2 about x^1 to

m in x^1y^1. Draw bb^2 parallel to ps and turn b^2 and a^2 about x^1 to meet x^1y^1 in n and k respectively.

Draw a^2A, bB, oO perpendicular to ps; and kA, nB, mO parallel to ps. Draw AB and Ap, and with O as centre, radius o^2v, draw the circular arc $CDEFGH$. Then $ABsp$ is the plane containing AB and the rays through A and B rabatted into HP about the horizontal trace sp, and the circle $CDEFGH$ is the section of the sphere by the plane

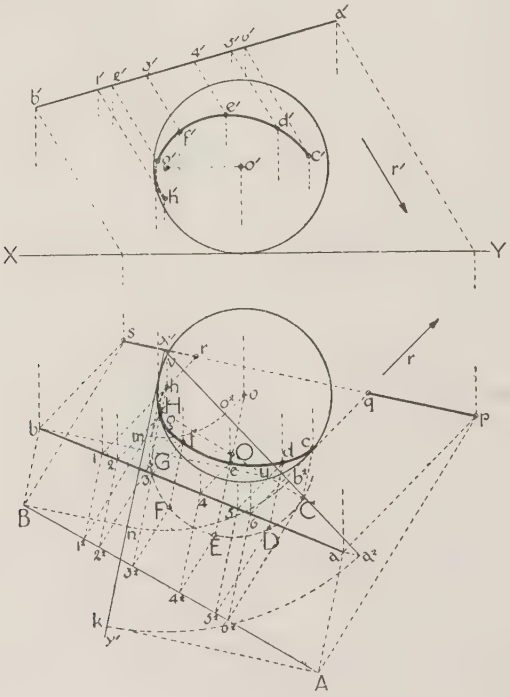


Fig. 322.*

rabatted into HP about sp, and Ap is the direction of the rays across the rabatted plane. Let 6^2C and 1^2H be tangents to the rabatted section at C and H respectively, parallel to Ap. Draw 6^26 parallel to Aa, $6q$ parallel to r, and Cc parallel to 6^26; then c is the plan of one extremity of the shadow on the sphere, and qp is the shadow of the part of the line AB, between 6^2A, which falls on HP. The plan of the second extremity h, and intermediate points d, e, f, g, are determined in a similar manner. The elevation of the shadow may now be determined by projection.

Shadows cast by Geometrical Solids.

Fig. 323.—The projections of a square prism, a cone, and a sphere are shown in the figure, and the shadows of the cone and sphere on the

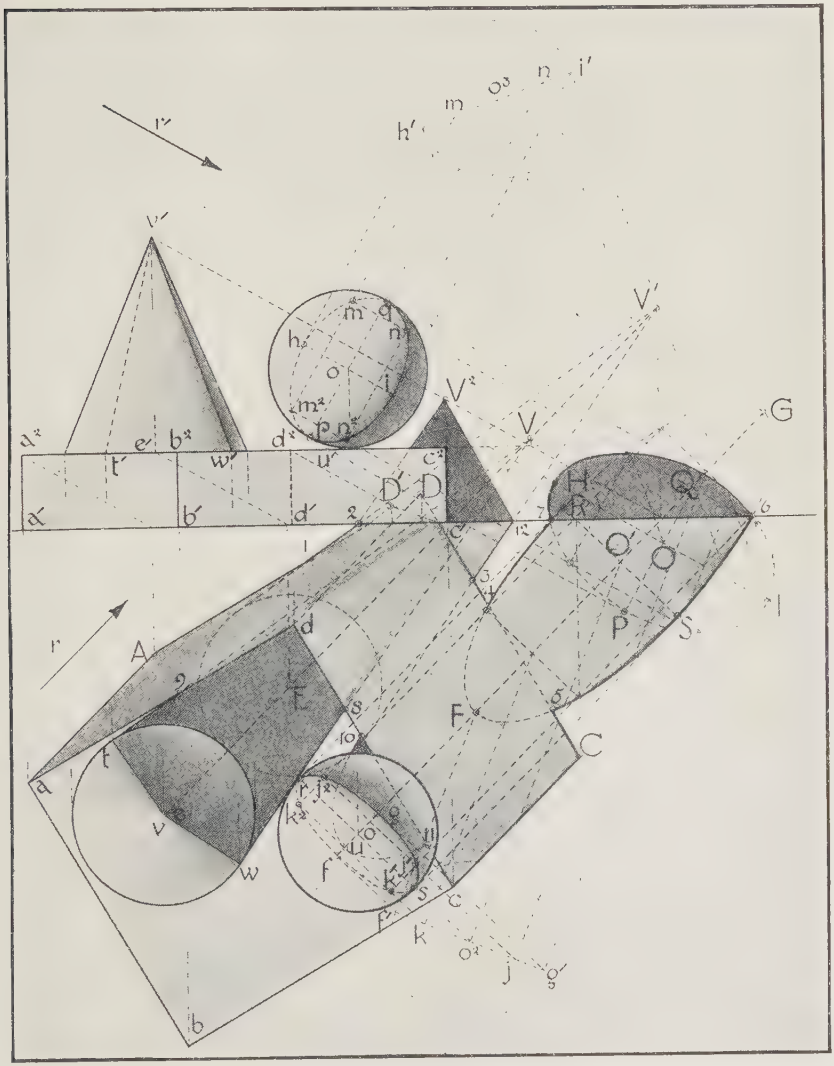


Fig. 323.*

prism, and of the three solids on the co-ordinate planes, together with the lines of separation between light and shade on the cone

and sphere, are required. r and r^1 indicate the direction of parallel light rays.

Consider the prism, plan $abcd$, elevation a^1, a^2, c^1, c^2, d^1, d^2. The horizontal traces of the rays that impinge on the upper angular points a, d, c of the prism are A, C, and D, D being in the second dihedral angle. Join C to D and A to D and draw aA and cC. D^1 is the vertical trace of the ray impinging on dd^2, and if the shadow cast by the prism alone were required, D^1 would be involved in the shadow on VP, as shown by dotted lines.

Now consider the cone. V^1 is the horizontal trace of the ray impinging on the vertex vv^1, and E is the horizontal trace of an imaginary ray through ee^1, the centre of the base ; and because the plane of the base is parallel to HP, the imaginary shadow of the base circle is an equal circle, centre E. Tangents to the latter circle from V^1 determine the shadow of the cone on HP, but only the sections of the tangents between 1, 2, and 3, 12, that lie in the first dihedral angle outside the prism shadow, are required in the shadow outline. V^2 is the vertical trace of the ray impinging on the vertex, and the elevation of the shadow of the cone on VP is $2V^212$, part of which is behind the elevation of the prism. V is the trace of the vertex ray on the plane containing the upper surface of the prism, and tangents to the plan of the base circle from V, cutting cd in 8 and ad in 9 and touching the plan of the base circle in t and w, give the shadow of the cone on the prism, and tv, vw, t^1v^1, v^1w^1 are the lines of separation on the conical surface.

The shadow of the sphere on the co-ordinate planes. If a right circular cylinder is arranged to envelop the sphere and have its axis parallel to the light rays, it will have elliptic lines of intersection with HP and VP, and the parts of the lines which fall in the first dihedral angle and are outside the prism and cone shadows will be the outlines of the shadow required, and the circle of contact between the sphere and cylinder will be the line of separation of light and shade on the sphere. The ellipses $GFRS$ and $HIQP$ are determined as explained in the chapter on cylinders, and the complete construction lines shown render further explanation unnecessary.

The shadow of the sphere on the prism is $10u11$, and this is equal to the elliptic arc $4F5$.

The Line of Separation between Light and Shade on the Inner Surface of a Spherical Dome with a Circular Opening in the Crown.

Fig. 324.—r, r^1 are the projections of one of the given parallel light rays ; R is the ray rabatted into HP.

First determine the arcs in the upper and lower arrises of the opening involved in producing the required line of separation. Let $h^2h^3r^2s^2$ be an elevation of the opening on hk as ground line, hk being through the plan, centre o, parallel to r.

Draw a perpendicular to hk through o to cut h^2r^2 in m^2 and h^3s^2 in x. Through u, the mid-point of m^2x, draw a^2b^2 parallel to R. Draw b^2bb^3 parallel to m^2o. Draw wi tangent to the inner base curve and parallel to R, and from i, the point of contact, draw a normal to cut

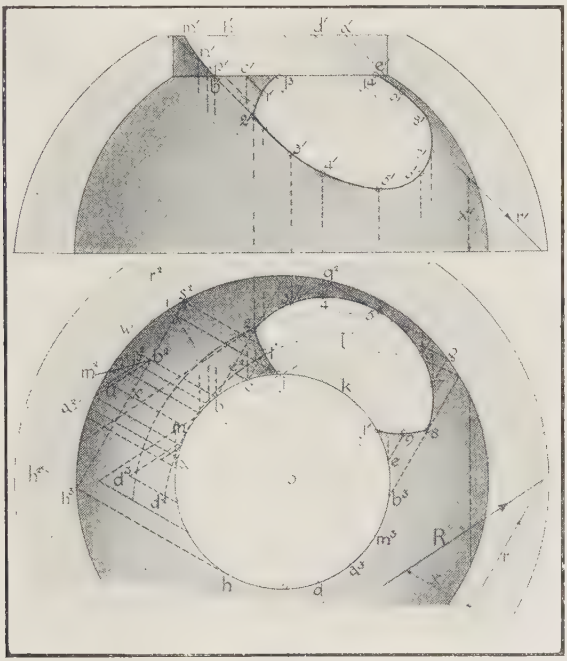


Fig. 324.*

h^3s^2 in j^2. Project j^2 to j and j^1, and a^2 to a and a^3. Then the lower arcs bj, b^3j^1 and the upper arc aha^3 have shadows falling on the inner spherical surface, these shadows being the required line of separation.

The shadows of the arcs am, a^3m^3 fall on the cylindrical surface of the opening as shown at $m^2n^2p^2b^2$ and $m^1n^1p^1b^1$. The points jj^1 are the plans of the points in which the line of separation meets the lower arris of the opening.

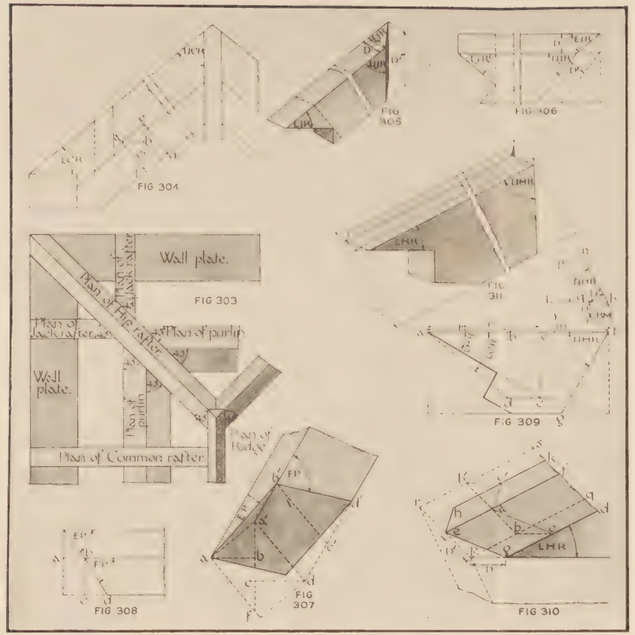
Consider the ray which impinges on the upper arris of the opening at h, h^1, h^2. The vertical plane containing the ray contains the centre of the sphere, and therefore the section of the inner surface of the sphere

by the plane is equal to the plan circle, and h^2h^3 is a section of the opening by the plane. Then h^2q^2 is the ray rabatted into the plane of the paper, and q^2 is an elevation of the point in which the ray meets the surface of the sphere, and since hk produced is the plan of the ray 5, the foot of the perpendicular to hk from q^2 is the plan of a point in the line of separation, and 5^1 is its elevation.

Now let gd^2 be the vertical section on de rabatted about de into the plane of the paper, and let d^2d^3 equal h^2h^3. Draw ec^2 perpendicular to de, and through d^3 and c^2 draw parallels to R to cut the arc gd^2 in l and f respectively. From l and f draw straight lines to cut de produced, perpendicularly in 7 and 9. Then 7 and 9, and 7^1 and 9^1, are the projections of two points in the line of separation ; and if c is the point of intersection between ee^2 and the plan circle of the opening, a parallel to de from c cuts the lines $9f$ and $7l$ in points 1 and 3 respectively, and $1\text{-}1^1$, $3\text{-}3^1$ are the projections of points in the required line of separation.

The ray impinging on the upper arris of the opening at aa^2 and on the lower arris at bb^2 meets the spherical surface in $2\text{-}2^1$, the point in which the shadows of the two arrises intersect ; $8\text{-}8^1$ is a similar point obtained from the ray a^3b^3.

PLATE XI.



FIGS. 303, 304, 305, 306, 307, 308. 309, 310, and 311.*

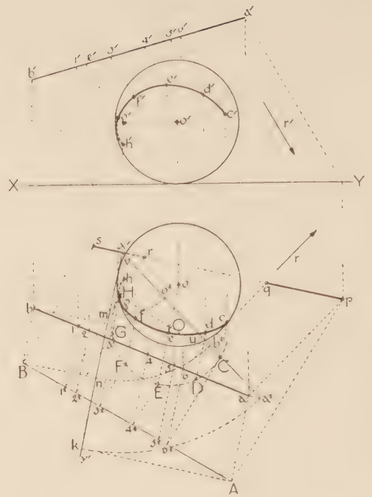


FIG. 322.*

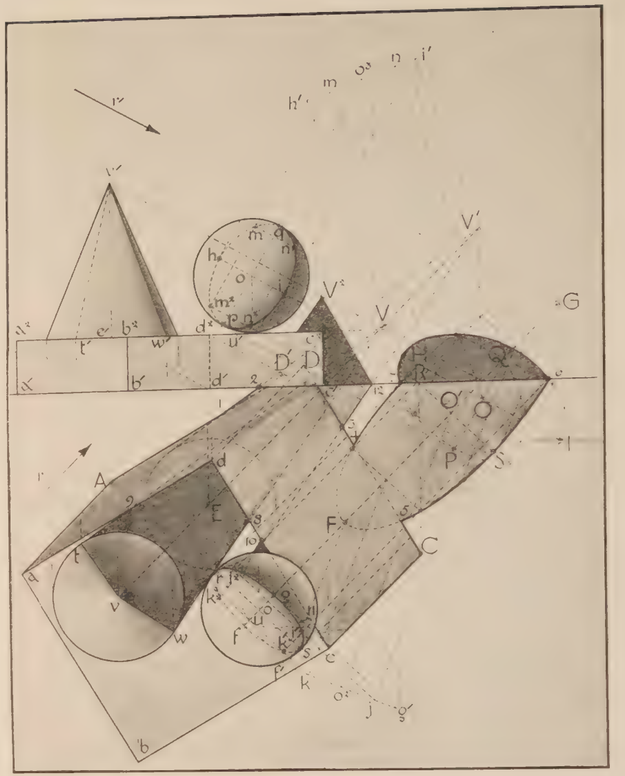


FIG. 323.*

see page 294.

CHAPTER XX

PERSPECTIVE PROJECTION

In perspective projection the lengths of the lines are governed by the angles between the light rays proceeding from the extremities of the corresponding lines in the actual object to the eye of the observer. Therefore the comparative lengths of the lines in the perspective drawing are controlled, not only by the comparative lengths of the corresponding lines in the actual object, but also by the distances of these lines from the observer's eye.

To obtain the appropriate lengths of lines at varying distances from the eye, an imaginary transparent plane, called the picture plane, may be supposed at any desired position between the object and the eye, and the distances apart of the points in which the light rays from considered points in the object intersect this plane are the required lengths.

The most easily understood method of bringing the appropriate lengths into their proper relative positions is illustrated in Fig. 325.

Let $ab \ldots h$, $a^1b^1 \ldots h^1$ be the projections of a rectangular building covered by a pitched roof, door and window openings being ignored for the present.

Let SP and SP^1 be the projections of the position of the spectator's eye, and let the straight line PP^1 be the edge view of the picture plane.

Draw straight lines from the plan points converging towards SP and meeting the picture plane in 1, 2 . . . 10, and from the elevation points converging towards SP^1 and meeting the picture plane in 1^1, $2^1 \ldots 10^1$. Then the horizontal distance between any two points in the perspective view will be equal to the distance between the intercepts of the rays from the corresponding points in plan on the picture plane. Thus the horizontal distance between the points aa^1, cc^1 will, in the perspective view, be equal to the distance between 8 and 10 on PP^1. The vertical distance between any two points in the perspective view will equal the distance between the intercepts of the rays from the corresponding points in elevation on the picture plane. Thus the

vertical distance between the points aa^1, bb^1 will, in the perspective view, be equal to the distance between the points 6^1 and 8^1 on PP^1.

At any convenient position draw RQ to make 45° with PP^1, and draw perpendiculars to PP^1 from 1, 2 . . . 10, to meet RQ in 1^2, 2^2 . . . 10^2. Draw parallels to PP^1 from the points in RQ to meet perpendiculars to the picture plane from corresponding points in the elevation rays; thus, let the perpendicular from 1^2 meet the horizontal from 1^1 in f^2; the perpendicular from 2^2 meet the horizontal from 2^1

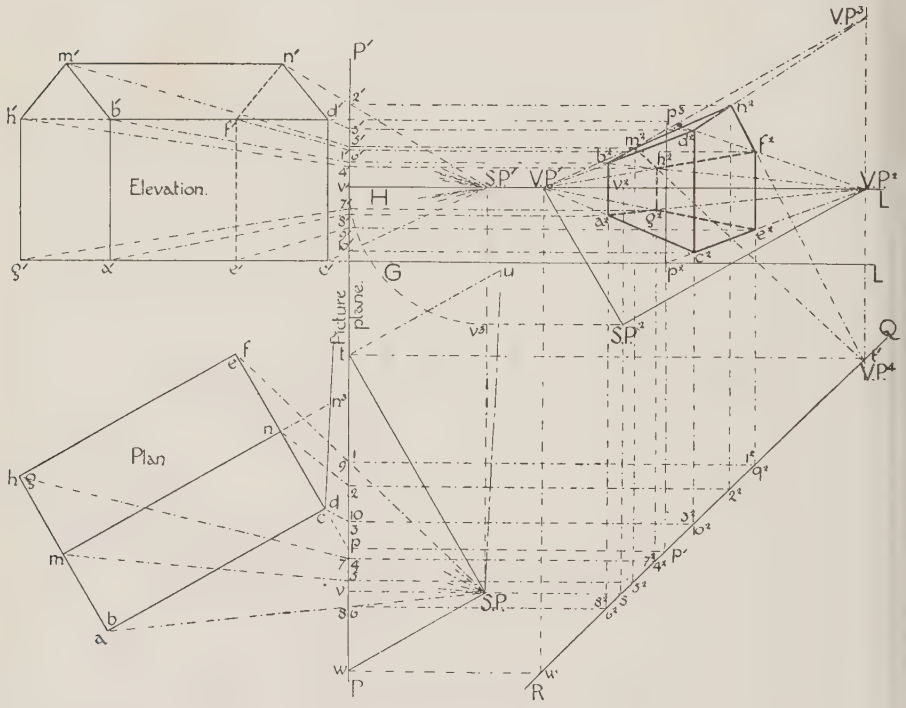


Fig. 325.*

in n^2, and similarly for the remaining points. Then, because 1 is in the plan ray from f and 1^1 is in the elevation ray from f^1, f^2 is the position of point F in the perspective view, and similarly n^2 is the position of point N, and f^2n^2 is the perspective view of the line FN.

When straight lines are drawn through the appropriate points in the perspective view, it will be found that all lines forming a parallel system in the actual object converge to a common point in the perspective projection. (This is only theoretically true for the vertical lines, but it will be shown that these conform to the general rule

governing perspective.) Each point of convergence is called a vanishing point, four such points being shown at VP^1, VP^2, VP^3, and VP^4; the position of an imaginary fifth point will be discussed later.

Refer now to the plan. Suppose the parallel straight lines through *ce* and *df* to be produced away from the picture plane to their imaginary point of intersection at infinity; then the light ray from this point to SP will be parallel to *ce* and *df*. Let SP, *t* be the plan of this ray between SP and the picture plane; then if *t* be brought into the perspective view in the same manner as were the other points, its position will be at VP^2; and since VP^2 is the perspective view of the imaginary point to which all lines parallel to CE and DF converge, it is the vanishing point in the perspective projection for all lines representing parallels to CE and DF.

Draw SP^1-v^1 perpendicular to PP^1, and with SP^1 as centre, turn v^1 to v^3. Erect a perpendicular from *s* to meet a horizontal from v^3 in SP^2, and join SP^2, VP^2. Project SP to *v* in PP^1, and SP^2 to v^2 in SP^1-VP^2; then, since SP^2-v^2 equals SP-*v*, and VP^2-v^2 equals *vt*, and the angles *tv*-SP, $SP^2v^2\,VP^2$ are right angles, the angle between the lines $SP^1\,VP^2$ and $SP^2\,VP^2$ is equal to the angle between the picture plane and the plane containing CE and DF.

It may be shown in the same way that the vanishing point for any system of parallel lines is the point of intersection between the picture plane and a straight line from the observer's eye parallel to the system; therefore VP^1, the vanishing point for all lines parallel to CA, is inclined to the horizontal line through VP^1 at an angle equal to the angle between the vertical plane containing CA and the picture plane.

Consider the vanishing points VP^3, VP^4 for the inclined lines d^2n^2, b^2m^2 and n^2f^2, m^2h^2. Make nn^3, in plan, equal to the vertical height of N above D, and join n^3d. Draw SP-*u* parallel to dn^3, and draw *tu* perpendicular to SP-*t*. Then SP-*u* may be regarded as the rabatment of either of the rays from SP parallel to FN or DN about the horizontal ray SP-*t*, and therefore VP^3 and VP^4 are in the perpendicular through VP^2 and the distance of each from VP^2 is equal to *tu*.

It will be seen that the vanishing points for all lines in a system of parallel planes are in the line of intersection between the picture plane and the plane containing the station point (observer's eye) parallel to the system. Thus, if VP^1 be joined by a straight line to VP^3, the angle between the straight line and the horizontal line HL will equal the inclination of the line of intersection between the picture plane and the plane through SP parallel to the plane of the roof surface *acmn*.

The vanishing points for lines parallel to the picture plane are, by the general law, at infinity; for a line from the station point parallel

to such lines will have its imaginary point of intersection with the picture plane at infinity. Therefore all lines in perspective projection representing lines parallel to the picture plane will be parallel.

Architectural Method of Perspective Drawing.

Fig. 326.—The usual method adopted by the architect in making perspective views is a slight modification of the method explained in the preceding example, the modification being such as to enable him to obtain his points from existing drawings or blue prints with a minimum of construction lines, as follows:

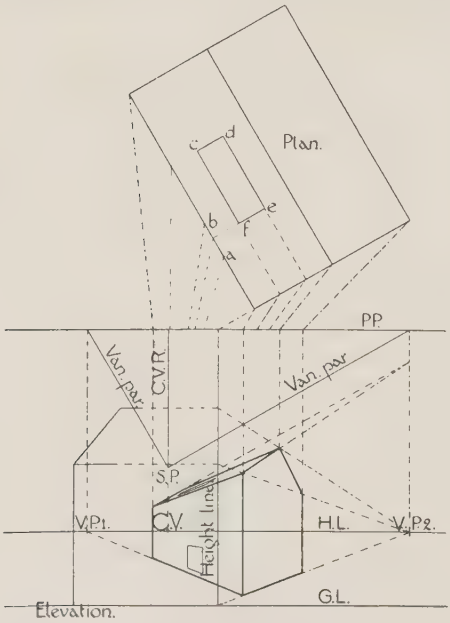


Fig. 326.

PP is the picture plane. The plan of the building, or of one floor of the building, is placed in the desired position behind *PP*. The station point, *SP*, is selected, and the vanishing parallels drawn parallel to the plan outlines. Lines converging to *SP* are drawn from all points in the plan to the picture plane. At any convenient position the horizontal line *HL* and the ground line *GL* are drawn parallel to *PP*, the distance between *HL* and *GL* being made equal to the required height of the station point. The elevation of the building or part elevation, as required, is drawn or pinned in a position similar to that shown. A plan boundary line is produced to *PP*, and a line is drawn perpendicular to the ground from the point of intersection to obtain the height line. The vanishing points are projected from *PP* to *HL*. Horizontals may now be drawn from the elevation to the height line, and from the points in the latter, lines to the vanishing points may be drawn. Projectors from the points in the picture plane determine the required points. A window opening and a skylight opening are shown in the figure, and the diagram should explain the construction for bringing them into the perspective view.

NOTE.—$C\,V$ is called the centre of vision; $C\,V\,R$, the centre visional ray.

School of Art Method of Perspective.

Fig. 327.—A knowledge of the use of measuring points and distance points in connection with perspective drawing is necessary to obtain the perspective view of an object without first making an orthographic projection.

Consider Fig. 327. S is the station point; S^1 the centre of vision; $S\,V$, $S\,V^1$ are the vanishing parallels and are inclined to the horizontal line at angles equal to the angles which the vertical planes in the object are desired to make with the picture plane, as already explained. Let ys equal the distance which the line nearest the picture plane is required to be to the right of the station point, and join S^1s. Make S^1D equal S^1S, and make sd equal the required distance of point A behind the picture plane. Draw Dd to cut S^1s in a, and join $a\,V$ and $a\,V^1$. With V^1 as centre, turn S to M^1, and with V as centre, turn S to M. Then M and M^1 are called measuring points. Draw straight lines from M and M^1 through a to meet the ground line in A^1 and A respectively. Draw $ABFCE$ to represent the lengths of the lines in the vertical end plane of the actual building to the scale adopted for the centre visional ray, and the height of the station point, and draw A^1F^1HP to represent the vertical side plane of the building to the same scale. Draw BM^1 to cut $a\,V^1$ in b, and PM to cut $a\,V$ in p. Draw $b\,V$ and $p\,V^1$ to intersect in q; then $abpq$ is the horizontal floor plane in perspective projection. Draw M^1E, M^1F, and erect perpendiculars from a and b to meet these lines in e and f. Draw $V^1\,V^4$ at right angles to $S\,V^1$ and make the angle $V^1S\,V^4$ (α) equal to the pitch of the roof (α^1). Make $V^1\,V^3$ and $V^1\,V^2$ each equal to $V^1\,V^4$; then V^2 and V^3 are the vanishing points for the roof planes.

NOTE.—c is in CM^1, h is in HM, and the combined use of the vanishing and measuring points provides alternative ways of obtaining the various points.

Let it be required to draw a semicircular headed door opening in the end wall, a rectangular window opening in the side wall, and a rectangular skylight opening in the roof, the sizes and positions of the openings being given.

Set out the semicircular opening $R\,U\,W\,T$ to the required size, and in the required position in the end wall $AECFB$. Draw the centre line $C\,X$, cutting the semicircle in 2, and draw perpendiculars from any number of points in the semicircle to meet the springing line $U\,W$ at 1, 1^2 and 3, 3^2. Draw straight lines from R, T, and X to M^1, to meet

ab in *r*, *t*, and *x* respectively. Erect perpendiculars from *r*, *t*, and *x*, and draw straight lines from *U*, *W*, and 2, to meet them in *u*, *w*, and 2^2 respectively. 1^1 and 3^1 are obtained in a similar manner.

Set out the rectangular window opening 8, 9, 10, 11 to the given

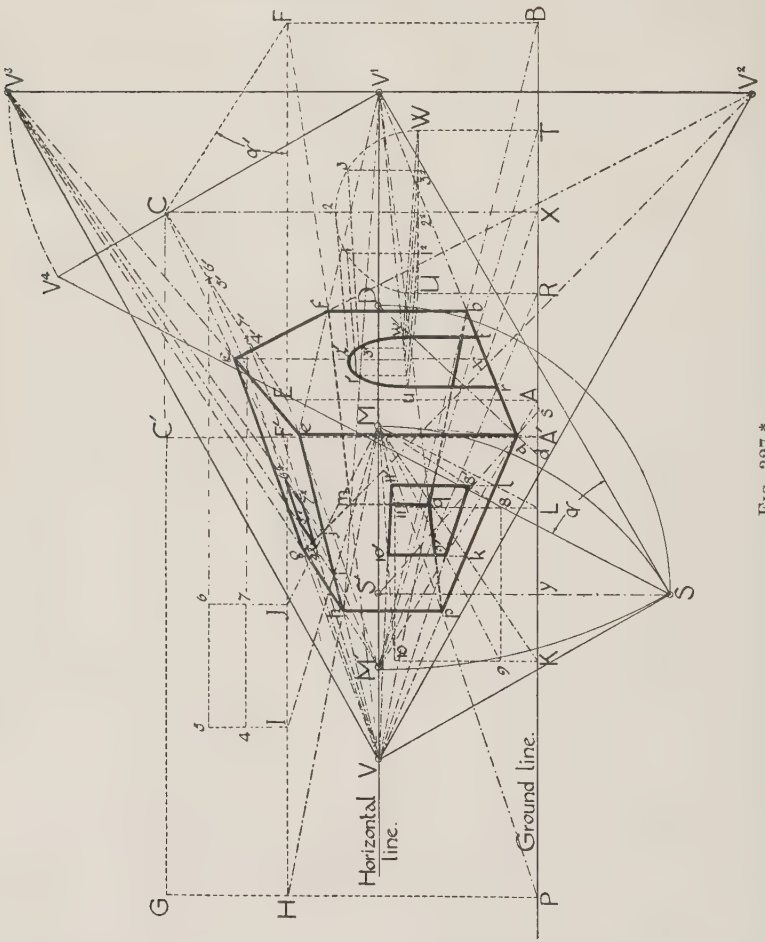


Fig. 327.*

size and in the given position in PHA^1F^1, and by drawing radials to *M* from *KL*, 8, 9, 10, 11, obtain *kl*, 8^1, 9^1, 10^1, 11^1.

Set out the given width of the skylight in the given position in *EC*, as 4^1, 5^1, and the given length *IJ* in the given position in HF^1; the elevation of the required opening is shown at 4, 5, 6, 7. Draw radials to M^1 from 4^1 and 5^1, and from their points of intersection with *ec* draw straight lines to *V*. Draw radials to *M* from *I* and *J*,

and from their points of intersection with *he* draw straight lines to V^3. The intersections of the straight lines to V and V^3 give the angular points of the perspective view of the opening.

If the student will track each line from the measuring points and vanishing points with a dry pen or similar pointer, he will obtain a clearer understanding of points and lines in the figure than can be gained from the most detailed explanation ; the full construction lines are included in the figure for that purpose.

Shadows in Perspective.

Fig. 328.—The figure comprises the perspectives of three solid steps, a vertical wall with inclined upper edge, and two square pillars. The perspective of the shadow cast by the complete solid on HP and of the shadows which fall on the solid are required, the direction of parallel light rays being given. Let R be the plan of the ray through the station point, and R^1 indicate the inclination of the rays to HP.

Produce R to meet the horizontal line in W, and produce R^1 through the measuring point M^1 to meet a perpendicular from W in W^1. Then W^1 is the vanishing point of the light rays.

First consider the shadow which falls on HP. Join a to W and 2 to W^1. Then $2W^1$ is the perspective of the ray impinging on the angular point 2, and aW is the perspective of the horizontal trace of the vertical plane containing the ray and the vertical arris $a2$; then 2^1, the point of intersection between the ray and the trace, is the perspective of the shadow of point 2, and $a2^1$ is the perspective of the shadow of the arris $a2$.

In this way it may be shown that the perspectives of the shadows of all vertical lines falling on HP vanish to W.

Produce 4, 3 to meet $a\,V^1$ in b. Join b to W and 3 and 4 each to W^1 to obtain 3^1 and 4^1, the perspectives of the shadows of 3 and 4 respectively. Then 3^1, 4^1 is the perspective of the shadow of the arris 3, 4, and 2^13^1 is the perspective of the shadow of the arris 2, 3, and because 2, 3 is horizontal and vanishes to V^1, and 2^1, 3^1 is the perspective of a line parallel to 2, 3, then 2^1, 3^1 also vanishes to V^1. It will be seen, then, that the perspectives of the shadows of horizontal lines on horizontal planes have the same vanishing points as the lines casting the shadows. Then 7^1z, the perspective of the line $7u$, vanishes to V.

The outline of the shadow cast by the upper end of the pillar 8, 9, 10 is determined in the same way, $z8^1$ vanishing to W, 8^19^1 to V^1,

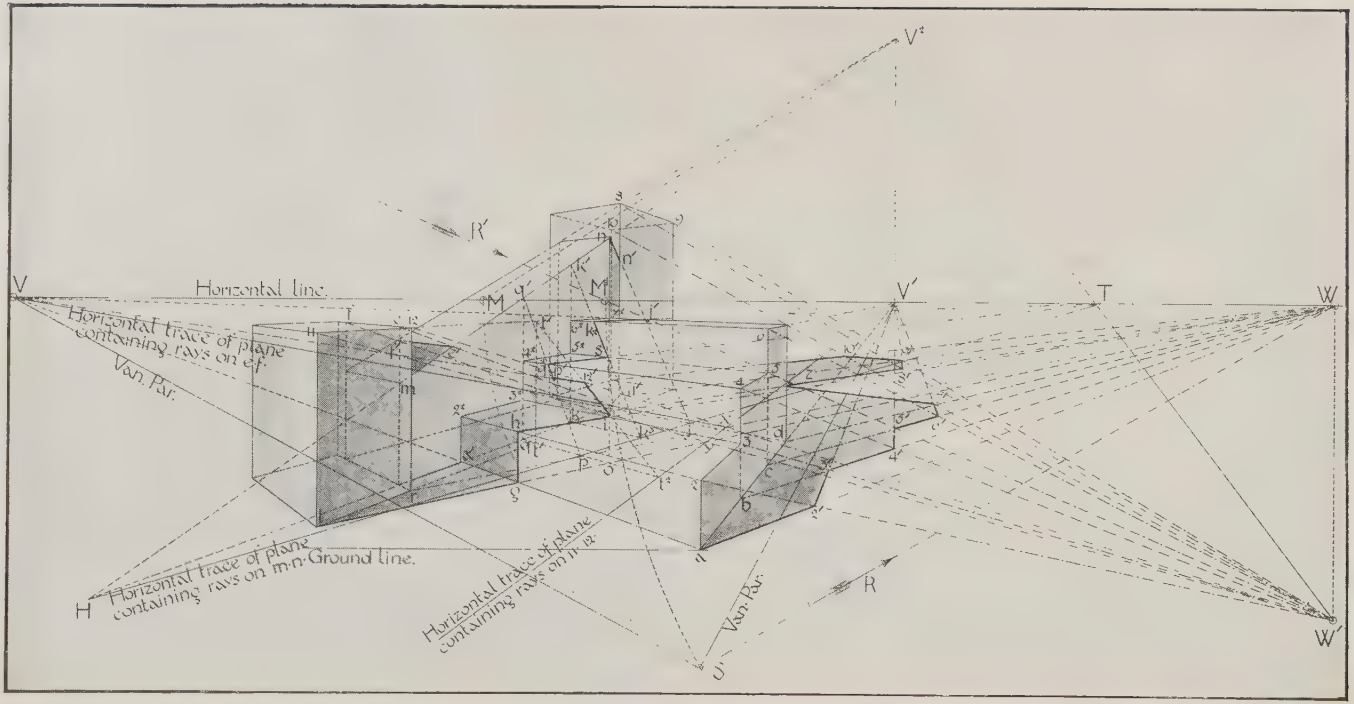

V²
R'
V
Horizontal line.
V'
T
W
Horizontal trace of plane containing rays on ef.
Van. Pt.
Horizontal trace of plane containing rays on m·n·
Ground line.
H
Horizontal trace of plane containing rays on rs
R
Van. Pt.
W'

FIG. 328.*

and $9^1 10^1$ to V; the remaining line, from 10^1, is directed to the foot of the perpendicular from 10.

Now consider the shadow of the forward pillar which falls partly on HP, partly on the steps, and partly on the inclined surface of the wall.

Draw lW to meet the lowest arris of the steps in g, $11W^1$ to intersect lW in y, and draw a straight line from V^1 through y. Then V^1y is the perspective of the horizontal trace of the plane containing the light rays that impinge on the arris 11, 12. Produce 3^2, 4^2 to q and q^1 and draw qt^2 (in the direction of V), t^2 being in V^1y produced, and join t^1t^2. Then t^1t^2 is in the plane containing the rays impinging along 11, 12, and also in the vertical plane of the riser 3, 3^2, 4, 4^2, and the points 11^1 and 12^1, in which $11W^1$ and $12W^1$ intersect t^1t^2, are the perspectives of the shadows of 11 and 12 on the riser face. Draw gh perpendicularly between aa^1 and 2, 2^2, and draw qW to meet 3, 3^2 in i. Draw $i11^1$, and note that $i11^1$ is in the perpendicular from o, o being the point of intersection between lW and tt^2; also that 12^1 is in the perpendicular from p, the point of intersection between rW and tt^2. Draw from 12^1 to V, to meet pq^1 in p^1. Then l, g, h, i, 11^1, 12^1, p^1 are in the perspective outline of the shadow of the pillar on HP and the steps.

From x, the point in which $12W^1$ meets V^1y, draw a straight line to V; then Vx is the perspective of the horizontal trace of the plane containing the rays impinging along the arris $12ef$, and e and f are in the vertical planes containing the wall surfaces. Draw straight lines from e and f to the points of intersection between Vx and the horizontal traces of the vertical wall planes, to cut the inclined wall arrises in e^1 and f^1. Then e^1f^1 is the perspective of the shadow of the arris containing ef across the inclined surface of the wall, and its vanishing point is V.

Erect a perpendicular from V^1 and draw a straight line from M to make an angle with the horizontal line equal to the actual inclination of the inclined wall arrises, and to meet the perpendicular in V^2; then V^2 is the vanishing point of the inclined arrises. Join V^2W^1 to cut the horizontal line in T. Produce nm to meet HP in H, and join HT. Then HT is the perspective of the horizontal trace of the plane containing all the rays impinging on mn, and the perspectives of any shadows cast by sections of mn on horizontal planes will vanish to T. Produce $5^2 6^2$ to k and k^1, and through k draw a straight line from V to meet HT in k^3. Draw k^1k^2 to meet 5, 5^2 in s and 6, 6^2 in k^2; then q^2s and k^2j^1 vanish to T, and n^1 may be determined by joining j^1n, since n^1 and j^1 are in nj, and nj is in the plane $HT\,V^2$ and the vertical plane containing the face of the pillar and the back of the steps.

PLATE XII.

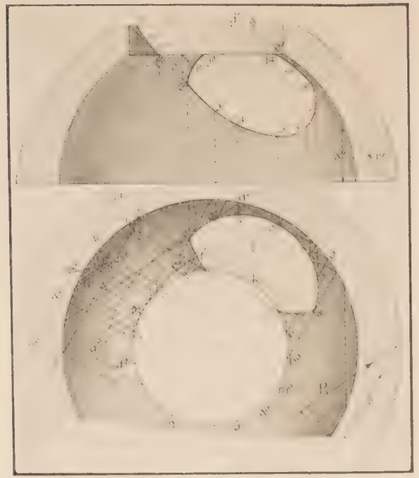


Fig. 324.*

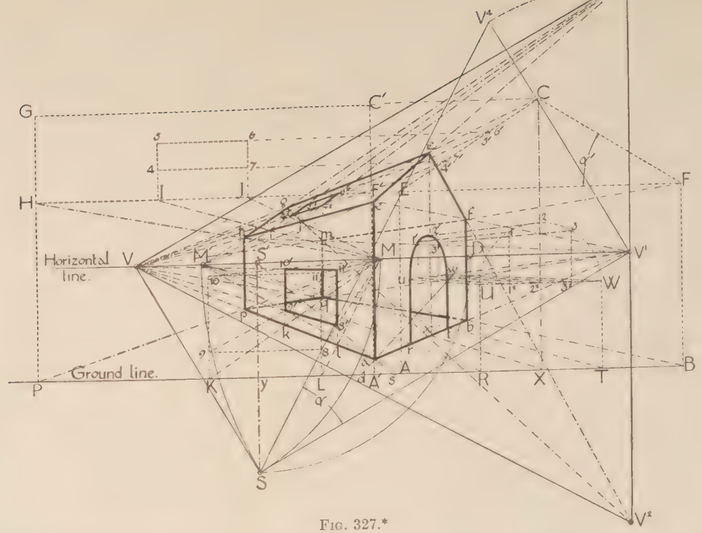


Fig. 327.*

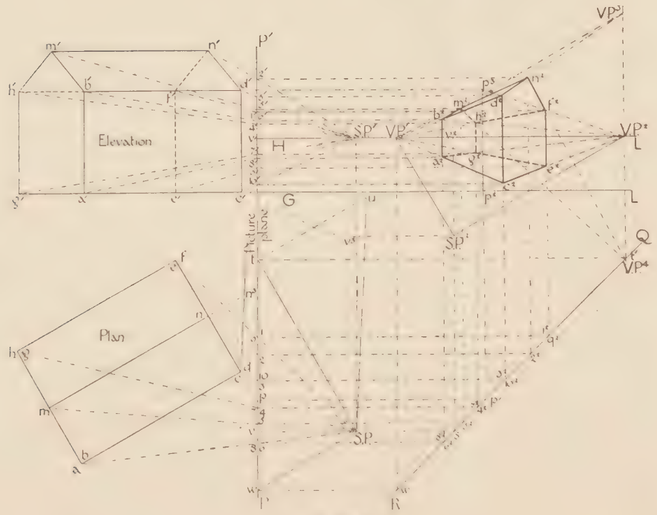


Fig. 325.*

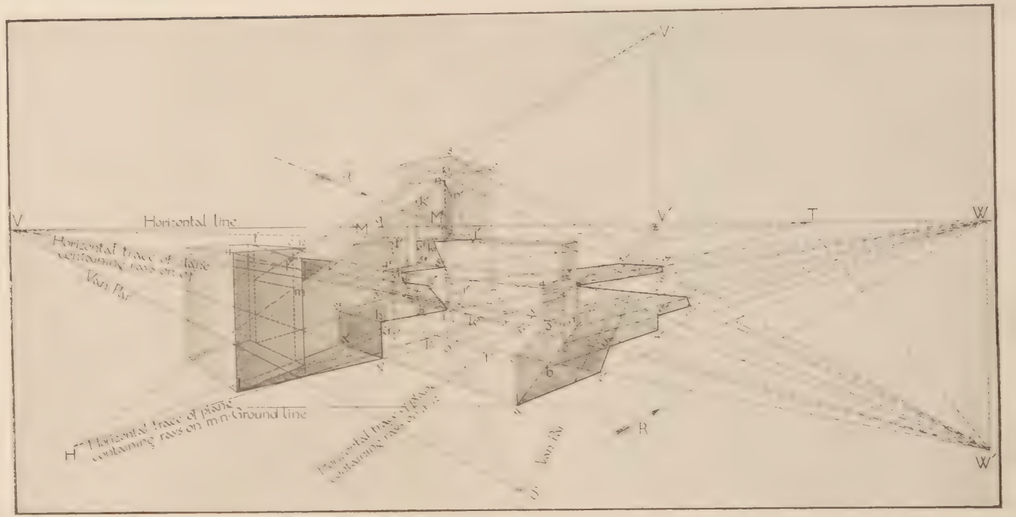


Fig. 328.*

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MISCELLANEOUS PROBLEMS

The following problems are collected from various sources. Some have been specially drawn up as having a direct bearing on particular Sections of the preceding Chapters, the others extracted from examination papers which are indicated by the following abbreviations.

Board of Education	B. o. E.
City and Guilds of London Institute . . .	C.G.
Union of Lancashire and Cheshire Institutes .	U.L.C.I.
Department of Agriculture, etc., Ireland . .	D.A.I.

Note to all figures.—All figures should be enlarged to any suitable dimension.

ON CHAPTERS I AND II

1. Make any point P on drawing paper. Draw a triangle ABA; AB 2 ins., BC 2·5 ins., AC 3 ins., such that the directions AP, BP, and CP are 30°, 45°, and 60° respectively east of north.

2. Draw any circle of 1·75 ins. radius, mark any point P 1 in. from the centre. Arrange a triangle similar to ABC in the preceding question with the angular point corresponding to A in the point P and the remaining angular points in the circumference of the circle.

3. Draw a circular arc of 3 ins. radius to subtend an angle of 57·3° and with the dividers step off three equal divisions along the arc. By construction shown in Fig. 6 (p. 7) develop the arc and the points of division into a straight line tangent and test the result with the dividers set to 1 in.

4. Set out the circular arc and the three equal divisions as in preceding question and apply the construction shown in Fig. 10 (p. 9). The figure must be drawn with the greatest possible accuracy to give an idea of approximation when trisecting the angle with this construction.

5. Draw three circles A, B, and C of radii 1 in., $\frac{3}{4}$ in., and $\frac{1}{2}$ in. respectively; distances between the centres of AB, BC, and CA 3 ins.,

$2\frac{1}{2}$ ins., and $3\frac{1}{2}$ ins. respectively. Determine a point P such that all tangents from P to the three circles are of equal length.

(Hint : The point may be found by drawing the radical axes.)

ON CHAPTERS III, IV, AND V

6. Draw a triangle ABC having a right angle at C, and, without drawing the circle, mark the points in which AB will be cut by the circle which passes through C and through the middle points of BC and CA. (B. o. E.)

7. (*a*) If a straight line drawn through S, one of the centres of similitude of two circles whose centres are C and C', cut one circle in P, Q, and the other in P', Q', show that the rectangles $SP.SQ'$ and $SQ.SP'$ are each equal to the rectangle contained by the tangents from S to the circles.

(*b*) Show how to describe a circle to pass through a given point and touch two given circles. (B. o. E.)

8. A circle is drawn within the angle BAC, formed by two straight lines AB, AC. Show how to draw the circles which touch AB and AC, and with which the given circle has internal contact. Give a carefully drawn diagram. (B. o. E.)

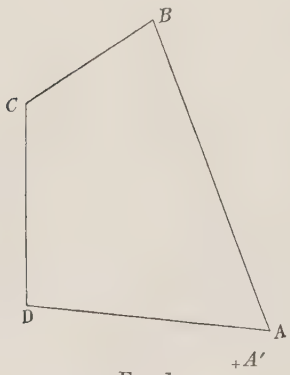


Fig. 1.

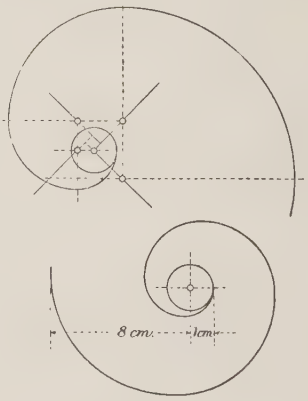


Fig. 2.

9. Define supplementary angles, and show that the opposite angles of any quadrilateral inscribed in a circle are supplementary.

Show also that the angles subtended at the centre of a circle by two opposite sides of a circumscribed quadrilateral are supplementary. (B. o. E.)

10. A and B are given points on the same side of a given straight line CD; show how to find a point P in CD so that the angle APB may be the greatest possible. (B. o. E.)

11. Two given circles touch each other internally; show how to draw a chord of the larger circle which shall touch the smaller circle and be of a given length.

When the diameters of the given circles are 4 ins. and 2 ins. respectively, construct a chord of the larger circle which shall be $2\frac{1}{2}$ ins. long and shall touch the smaller circle. (B. o. E.)

12. Fig. 1. Determine the point O about which the quadrilateral $ABCD$ can be turned through a clockwise angle of 55°, bringing A to A'. Draw the quadrilateral in its new position. (B. o. E.)

13. Fig. 2 shows a construction for describing a volute of one convolution. Set out a similar curve of one and a half convolutions, when the dimensions are as figured in centimetres. (B. o. E.)

(N.B.—Do not copy the diagram.)

ON CHAPTER VI

14. Two lines, 4 ins. and 2·5 ins. long, bisecting one another at 60°, are conjugate diameters of an ellipse. Plot the curve and draw its major and minor axes. (B. o. E.)

15. Draw, to a scale of 1 inch to a foot, the elevation of an elliptical arch, 6 ins. upon the face, for a 4-ft. opening having a rise of 1 ft. 6 ins., and showing the bed joints as true normals to the curve. State clearly how, in practice, the outer curve of the arch should be drawn, and the reasons for adopting this method. (C.G.)

16. Draw, to a scale of 1 inch to a foot, using the compasses, an elliptical arch, 12 ins. upon the face, for a 3-ft. opening. What method of setting out would you adopt if you were preparing for practical work, and how would you draw the extrados? (C.G.)

17. Tangents TP, TQ are drawn to a parabola from a point T. Show that they subtend equal angles at the focus, and have equal projections on the directrix.

Given S, the focus of a parabola, and a line PT touching the parabola at P, find the vertex of the parabola. (B. o. E.)

18. Show that in a conic section the sum of the focal distances of a point is constant if the eccentricity is less than unity, and the difference of the focal distances is constant if the eccentricity is greater than unity.

Two ellipses whose axes are equal are placed touching each other at extremities of their minor axes. Show that if one ellipse roll on the other, the foci of the rolling ellipse will describe circles having the foci of the fixed ellipse as centres. (B. o. E.)

19. Show that the exterior angle between any two tangents to a parabola is equal to the angle which either of them subtends at the focus.

A fixed tangent to a parabola cuts the tangents at the ends of any focal chord in P and Q; show that PSQ is a right angle, S being the focus. (B. o. E.)

20. Prove that in an ellipse the tangent at any point makes equal angles with the focal distances of the point of contact.

If a line drawn through the centre of an ellipse parallel to the tangent at any point P intersects the lines joining P to the foci in points Q and R, show that PQ and PR are each equal to the semi-major axis. (B. o. E.)

21. If a line touch a hyperbola at a point P, and meet the asymptotes at Q and Q', show that $PQ = PQ'$.

If the normal at P to a hyperbola meet the axes at G and G' respectively, and the tangent at P meet the asymptotes at Q and Q', show that the circle which passes through the centre of the hyperbola and the points Q and Q' also passes through G and G'. (B. o. E.)

22. Give briefly constructions for finding a number of points (i) on a parabola of given latus rectum, (ii) on an ellipse, major axis $2a$, minor axis $2b$.

Given the focus and two points on a parabola, show how to find the position of the vertex. (B. o. E.)

23. Draw by intersecting lines the outline of a parabolic arch, span 5 ft., rise 4 ft. Determine the following positions and state their relations to the curve and to each other: Directrix, Focus, Latus Rectum, Sub-normal, and Tangent. Scale $\frac{1}{2}$ inch = 1 foot.

24. If the tangent at any point P of a conic section whose focus is S meets the directrix at K, show that the angle PSK is a right angle.

In the case of a parabola, if PS produced meet the parabola again at P' and if a line through K parallel to the axis meet PP' in V, show that V bisects PP'. (B. o. E.)

25. Show how to draw tangents to an ellipse from an external point.

If the pair of tangents are at right angles, show that the locus of their point of intersection is a circle. (B. o. E.)

ON CHAPTERS VII, VIII, IX, X, AND XI

26. How is the angle contained by two planes measured ?

If a plane contain a line which is perpendicular to a second plane, show that the two planes are at right angles to each other.

OA, *OB* are two intersecting straight lines ; *OP* is a third straight line equally inclined to *OA* and *OB*, but not in the plane *AOB*. Show that the projection of *OP* on the plane *AOB* bisects the angle *AOB*. (B. o. E.)

27. Draw the traces of a plane inclined at 45° to the horizontal and perpendicular to the vertical plane of projection. Two lines in this plane make angles of 30° and 60° with a line of greatest slope on opposite sides of it. Find the inclinations of these lines to the horizontal, and the angle between their horizontal projections. (B. o. E.)

28. *VO*, *OH* are the traces of an oblique plane ; *P* is a point in the line of intersection of the planes of projection. Draw the projections of a perpendicular *PM* on the plane. Find the lengths of *PM* and *OM*, and the angle between *OP* and the plane *VOH*. (B. o. E.)

29. Two lines *OA*, *OB*, include an angle γ, a third line *OC* makes an angle α with *OA*, and β with *OB*. Find the angle which *OC* makes with the plane *AOB*, and also the angle between the planes *AOC* and *BOC*. (B. o. E.)

30. Construct the projections of a line which passes through a point 2 inches from each plane of projection, and which is inclined 30° to the horizontal plane and 40° to the vertical plane. Also determine the traces of a plane passing through this line, and inclined 50° to the horizontal plane. (B. o. E.)

31. A cube of 2 ins. edge has one of its faces inclined 35° to the horizontal plane, and an edge of this face inclined 20° to the same plane. Draw a plan and elevation of the cube. (B. o. E.)

32. Two planes *ABC*, *ABD*, which include an angle of 80°, are cut by a third plane in lines *AP*, *AQ*, which are inclined at 50° and 70° respectively to the line *AB*. Find the angle *PAQ*, and also the inclination of the third plane to each of the given planes. (B. o. E.)

33. *ABD* is a triangle in the vertical plane of projection, having each side 3 inches long, and *ABC* is an isosceles triangle in the horizontal plane, having the angle *C* a right angle. Construct the projections of a line which passes through the middle point of *CD*, and cuts at 50° the line whose vertical and horizontal projections lie along *DB* and *AC* respectively. (B. o. E.)

34. The horizontal and vertical traces of a plane are inclined at 40° and 50° to the ground line. Construct the real angle between its traces, and the angle at which the plane is inclined to the horizontal.

Find also the projections of a line which is perpendicular to the given plane, and passes through a point distant 1 inch from each plane of projection, and also distant 1 inch from the given plane. (B. o. E.)

35. A thin plate is cut to the shape of a parallelogram *ABCD*, sides = $3\frac{1}{2}$ ins. and $2\frac{1}{2}$ ins., angle $BAD = 60°$. The plate is then bent to a right angle along the diagonal *AC*, thus forming a geometrical model.

(*a*) Draw the plan of the model when it rests with its edges *CB*, *CD* on the ground.

(*b*) Find the horizontal traces of the planes which bisect the angles *BAC*, *DAC* at right angles to their planes.

(*c*) Determine the axis and the horizontal trace of a cone of revolution which has the lines *AB*, *AC*, *AD* on its surface. (B. o. E.)

36. The roof of a house is rectangular in plan, and two adjacent surfaces are inclined to the horizontal at 30° and 40° respectively. Find the inclination to the horizontal of the "hip," or line in which the two surfaces meet.

What is the magnitude of the dihedral angle between the surfaces? (B. o. E.)

37. Draw the projections of a line 4 ins. long, inclined at 30° to the horizontal, and at 45° to the vertical plane of projection, and having one end in the horizontal plane 3 ins. in front of the ground line. Find the traces of a plane bisecting this line and perpendicular to it. (B. o. E.)

38. Two planes contain an angle of 64°; one of them is inclined to the horizontal at 52° and their intersection is inclined at 40°. Represent the planes by their traces. (B. o. E.)

39. *OP*, *OQ* are the traces of a given plane which contains the base of a tetrahedron.

Having given the plan *ABC* of the triangular base, also the projections *V*, *V'*, of the vertex, construct one of the plane angles at the vertex, and one of the dihedral angles. Draw any suitable plan, *ABC*, and any suitable point, *V*,*V'*. (B. o. E.)

40. *P* is a point in the vertical plane of projection 3 ins. above the horizontal plane. Two lines passing through *P* include an angle of 40°, and are inclined 50° and 65° respectively to the horizontal plane.

Construct the projections of these lines, the traces of the plane containing them, and the angle at which this plane is inclined to the horizontal. (B. o. E.)

41. Construct the traces of a plane which is inclined at 65° to the horizontal plane, and at 40° to the vertical plane.

Find the projections of a point in this plane distant 2 ins. from the vertical plane, and $1\frac{1}{2}$ ins. from the horizontal plane.

A line 3 ins. long is bisected at right angles by the plane at this point; construct the projections of this line. (B. o. E.)

42. A level rectangular tennis lawn, 40 yards by 20 yards, bounded by cuttings and embankments, is to be made on sloping ground by the transference of soil from the higher to the lower levels.

Inclination of original surface of ground 10°. Direction of lines of steepest slope and of short edges of lawn, East and West. Inclination of surfaces of cuttings and embankments, 45°.

Draw a plan of the lawn, showing the boundaries of the finished earthwork. Scale, $\frac{1}{8}$-in. = 1 yard. (B. o. E.)

43. Two inclined faces of rock A,B in a quarry intersect in L. The lines of steepest ascent on A make 10° with the vertical and go due North. Those on B make 20° and have a direction 12° North of East. A seam of coal intersects these faces in lines which both go upwards from L, making 70° with L on the face A and 65° with L on the face B. Find the direction, and inclination to the horizontal, of the lines of steepest descent of the seam. (B. o. E.)

44. Two planes make an angle of 90° with each other, both being equally inclined to the horizontal plane; the angle between their traces is 75°. What will be the angle each makes with the horizontal plane? (C.G.)

45. A person using a theodolite at a station point A observes the angular elevations of two objects P and Q above the horizon to be 35° and 65° respectively. The horizontal angle (or azimuth) between their directions, that is, between vertical planes containing them, measures 75°. Find the angle PAQ subtended by the two objects at the place A, as would be measured by a sextant. (B. o. E.)

46. Three lines OA, OB, OC in space are mutually inclined to one another at 60°. A line OP, situated within the trihedral angle $OABC$, makes 50° with OA and 40° with OB. What is the angle POC? What angle does OP make with the plane OAB? (B. o. E.)

47. The positions of two points P and Q in a room are specified

in the following table, where x, y, and z are distances in feet from a side wall, an end wall, and above the floor respectively :

	x	y	z
P . . .	1	4	2
Q . . .	2·7	1·6	5·5

Draw, to a scale of $\frac{1}{2}$ in. to 1 foot, a figured plan showing the faces of the walls and the two given points. Find and measure the distance from P to Q, and the angles which PQ makes with the floor and the walls. (B. o. E.)

47A. The vertical trace of a plane makes an angle of 25° with the ground line, and its horizontal trace an angle of 50°. A point P is $\frac{1}{2}$ in. distant from the plane on its upper side, 2 ins. in front of the vertical plane and 3 ins. above the horizontal plane. A sphere of diameter 3 ins. has its centre at P. Draw the projections of the line of intersection of the plane and sphere. (B. o. E.)

48. A right circular cylinder, whose diameter is $2\frac{1}{2}$ ins., lies on the horizontal plane of projection, and its axis makes an angle of 50° with the vertical plane of projection. Construct the traces of a plane touching the cylinder and making an angle of 40° with the horizontal plane. Also construct and measure the angle which this tangent plane makes with the vertical plane of projection. (B. o. E.)

49. Two planes are given by their traces, Fig. 3. Show their intersection. Find the dihedral angle between them. Determine the

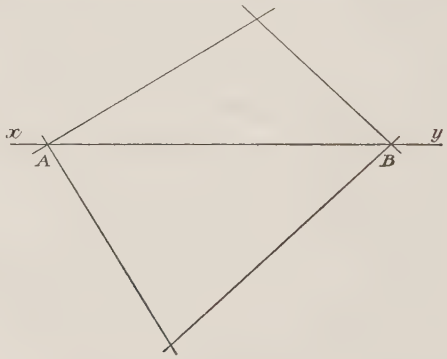


FIG. 3.

projections of the shortest path on the planes going from A to B. What is the length of this path ? (B. o. E.)

50. Fig. 4 is the plan of an embankment rising from level ground and level on the top. The side A has a slope of 40° to the horizontal; what is the slope of the side B?

The lines pp, pp represent the sides of a road cut through the embankment, the slopes of the cutting being 35° on each side of the road. Complete the plan of the cutting. (B. o. E.)

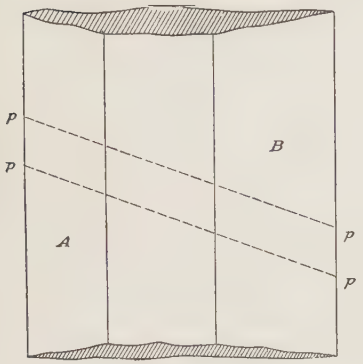


Fig. 4.

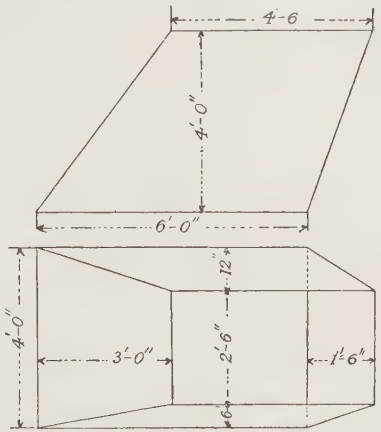


Fig. 5.

51. Fig. 5 shows the plan and elevation of part of a smoke uptake. Draw to a scale of 1 in. to the foot the pattern for each of the sides. Find also the true bevel or angle to which the angle irons forming the corners must be set to suit. (C.G.)

52. Fig. 6 shows the plan and elevation of an elbow or knee on a square pipe. Draw the required patterns for the elbow, each face of the pipe being in one piece. Scale 1 inch = 1 foot. (C.G.)

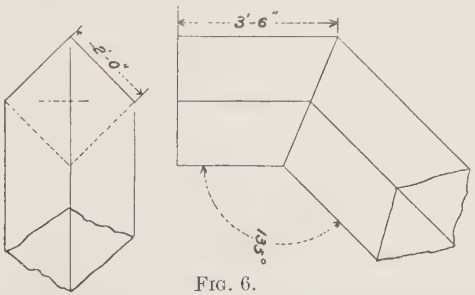


Fig. 6.

53. Two projections of a prismatoid are given in Fig. 7. Show the development, in a single piece, of one of the symmetrical halves of the sloping surface.

Draw the plan of the mid-section SS, determine its area a and the areas a_1 and a_2 of the ends of the solid. Measure the height and calculate the volume of the solid by using the prismoidal formula

$$\text{volume} = \text{height} \times \frac{a_1 + 4a + a_2}{6}. \qquad \text{(B. o. E.)}$$

54. A geometrical model is made by fixing a piece of bent wire ABC in a board as shown in Fig. 8.

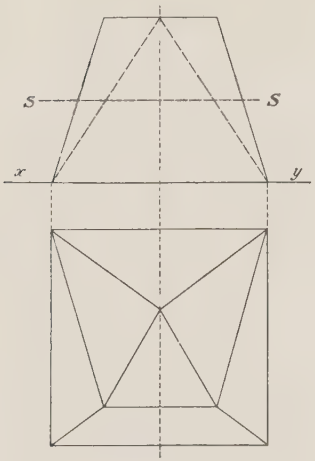


Fig. 7.

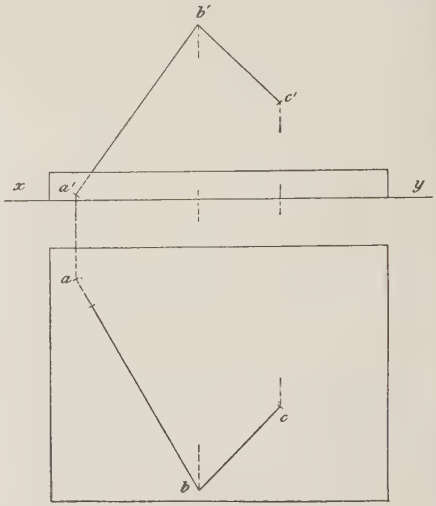


Fig. 8.

(*a*) What is the true angle between the wires at B ?

(*b*) Show the projections of the conical surface that is generated by BC when the wire AB is turned about its axis. (B. o. E.)

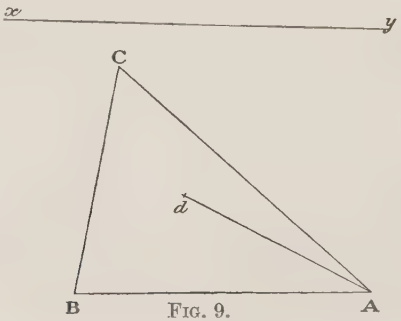


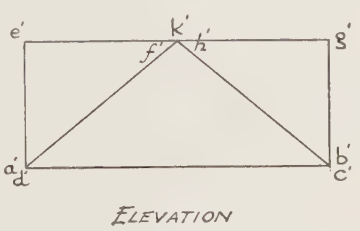
Fig. 9.

55. A thin metal plate ABC resting on the ground is shown in plan Fig. 9. A piece of thin wire AD, 2·5 ins. long, with one end soldered to the plate at A, is also shown in plan at Ad.

(*a*) Find the distance of the end *D* of the wire from the plate, and draw the elevation of the wire and plate on *xy*.

(*b*) Draw the plan of a sphere which touches the plate, has its centre *O* on *AD*, and which passes through *D*.

(*c*) Determine a plane which contains *BC* and touches the sphere. (B. o. E.)



56. In Fig. 10 the plan and elevation of a stone cap are given. Determine the real angle between the surfaces *A* and *B*. (D.A.I.)

57. Draw to a scale of 3 ins. to 1 foot the correct shape of a piece of sheet-lead to make the 4 in. by 3 in. rectangular bend shown in sketch (Fig. 11), it being understood that the edges of the lead will be joined by a burnt seam at sides of pipe and throat of bend. (C.G.)

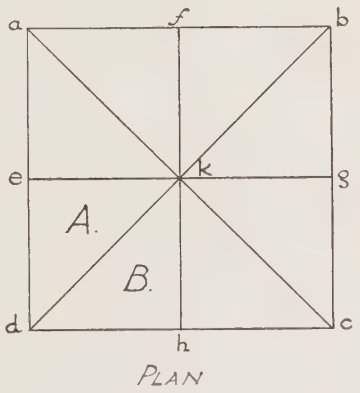


Fig. 10.

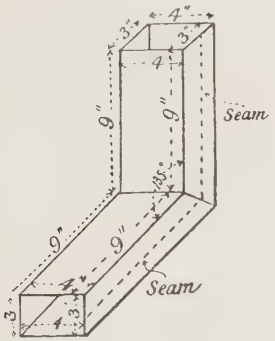


Fig. 11.

ON CHAPTER XII

58. Define a Sphere, and explain the terms Great Circle, Small Circle, Poles of a Circle. (B. o. E.)

59. Having given two points *P*, *Q*, on a given sphere, state in how many ways it is possible in general to join them by an arc of a great circle. Explain the exceptional case. (B. o. E.)

60. The three plane angles of a trihedral angle are 30°, 40°, and 50° respectively. Construct the three dihedral angles. (B. o. E.)

61. The three dihedral angles of a trihedral angle are 110°, 135°, and 155°. Construct the three face angles of the trihedral angle. (B. o. E.)

62. In Fig. 12 *dABc* is the plan of a piece of wire bent into three equal straight lengths, of which *AB* rests on the ground and *AD*, *BC* are inclined upwards—

(*a*) Determine the heights of *C* and *D*.

(*b*) Show the horizontal traces of the three planes which bisect the lines *AB*, *BC*, and *AD*, at right angles.

(*c*) The four points *A*, *B*, *C*, *D* determine a sphere ; draw its plan. (B. o. E.)

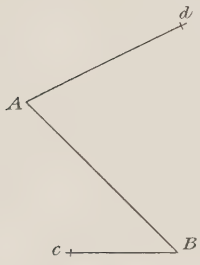


Fig. 12.

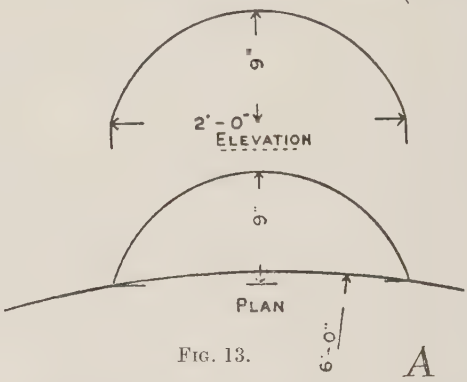


Fig. 13.

63. A segment-headed niche, segment in plan, is fixed in a circular wall all prepared for the plasterers' work (Fig. 13). Give the shape of the various ribs and their bevels. Scale 2 ins. = 1 foot. (C.G.)

64. A sphere, 2 ins. in diameter, has its centre 3 ins. in front of the vertical plane and 2 ins. above the horizontal plane. Construct the traces of a plane touching the sphere and inclined 50° to each plane of projection. Construct also the traces of two other tangent planes to the sphere, perpendicular to the first one, and to one another. (B. o. E.)

65. A square chamber of 10 ft. internal dimension, is covered by a spherical roof, the diameter of the sphere equalling the diagonal of the chamber. Draw a vertical section through the centre of the dome, parallel to one side of the chamber. Show the intersection of the domical roof and the face of one wall, and determine in the same view elevations of one of the longest, one of the shortest, and one other rib. All the ribs lie in radial planes. Show a circular opening 3 ft. diameter in the top of the dome. Size of curb round opening 4 ins. wide, 3 ins.

deep. The under edge of each rib dies out at the under and inner arris of the curb, and the upper edge of rib finishes 1 in. below upper surface of curb. Thickness of rib $1\frac{1}{2}$ ins. Scale $\frac{1}{2}$ in. to the foot.

(U.L.C.I.)

ON CHAPTER XIII

66. A sphere of 3 ins. diameter has its centre $2\frac{1}{2}$ ins. above the horizontal plane of projection; the axis of a cylinder enveloping the sphere is inclined at 45° to this plane. Construct the line of contact of the two surfaces, and the horizontal trace of the cylinder. (B. o. E.)

67. Represent a cylinder, $\frac{3}{4}$ in. diameter, $1\frac{1}{2}$ ins. long, standing upright on the horizontal plane, and having a helix of $1\frac{1}{2}$ ins. pitch traced on its surface.

Set out the horizontal trace of the surface which is formed by a line moving so as to be always tangential to the helix. Develop that portion of the surface which is contained between the extended end planes of the cylinder. (B. o. E.)

68. Fig. 14 shows an arch ring of seven stones, built to pass obliquely through a wall. (The wall is omitted.) Project an elevation

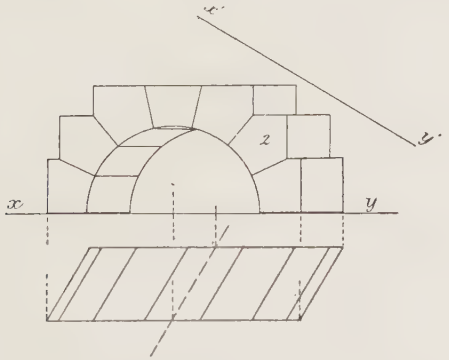


Fig. 14.

on $x'y'$. Develop the inner curved surface or soffit, showing the coursing lines. Draw a development of the second ring stone, enlarged to double size, thus exhibiting the shapes of the templates for its seven faces. Find the two dihedral angles between the front face and the two radial coursing joints of this stone. (B. o. E.)

69. The body-range of a semi-circular vault is 10 ft. wide. It is penetrated by the soffit lining of a semi-headed window 4 ft. in diameter,

both springing lines being in the same plane. Give the plan of interpenetration and the development of the lining. (C.G.)

70. The sketch, Fig. 15, is the plan of a chamber covered by brick vaulting. Make drawing, showing how the centring is made, and provision for striking the same. All geometrical work to be shown. (C.G.)

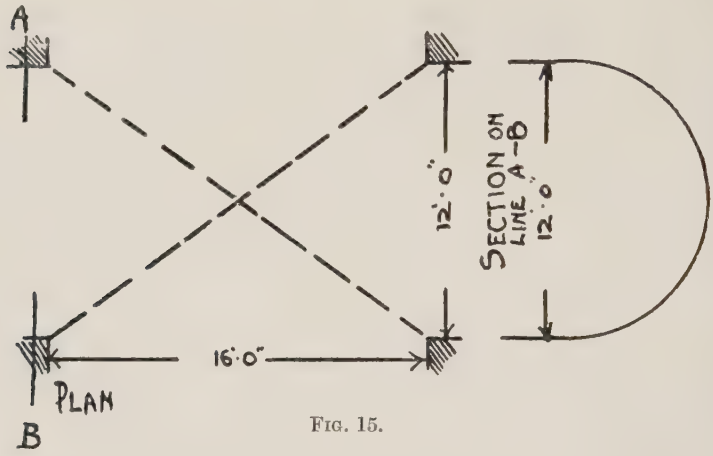


Fig. 15.

71. A semi-circular-headed vault runs into a circular chamber, with a spherical roof as shown in Fig. 16. Determine the plan and an elevation, at right angles to the vault, of the intersection between vault and dome. Scale $\frac{1}{4}$ inch = 1 foot. (U.L.C.I.)

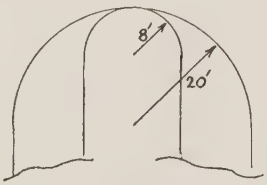


Fig. 16.

72. Determine the projections of the axis of a right circular cylinder, having given the radius, the projections of two points on the surface, and the projections of a generator.

Draw the traces of the plane touching the surface along the given generator. (B. o. E.)

73. A skew arch runs at an angle of 120° with the face of the outer walls. The arch, on the face of the wall, is semi-elliptical, span

20 ft., rise 6 ft. Draw an elevation in a direction at right angles to the face of the arch, and develop the whole of the arch soffit. Length of springing line 12 feet. Scale $\frac{1}{8}$ inch = 1 foot. (U.L.C.I.)

74. Fig. 17 gives the plan of the coping over a 9 ins. wall, the top of the coping sloping uniformly downwards from *A*, so that the difference in levels between *A* and *B* is 4 feet. The edges of the coping are cylindrical, and radial lines drawn across the upper and lower surfaces are horizontal.

To a scale of $\frac{1}{2}$ inch to the foot, draw an elevation in the direction *E*. Show the joint lines at right angles to the developed inner edge and project back into elevation. (U.L.C.I.)

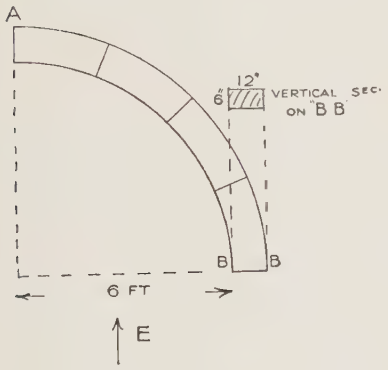


Fig. 17.

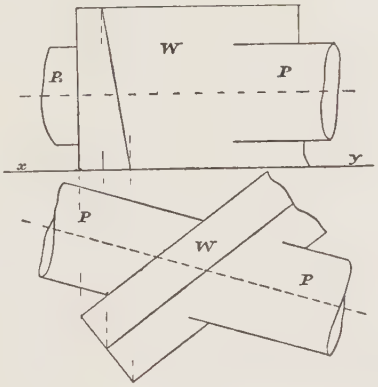


Fig. 18.

75. Show the hole in the wall *W* which will just allow the cylindrical pipe *PP* to pass through it. (See Fig. 18.) Set out templates suitable for marking the curves on the faces of the wall, drawing on each template a line which must be set horizontally. (B. o. E.)

ON CHAPTER XIV

75A. The axis of a right circular cone is inclined 30° to the horizontal and 40° to the vertical plane of projection ; the base is 3 ins. in diameter, and the altitude is 4 ins. Construct the plan and elevation of the cone. (B. o. E.)

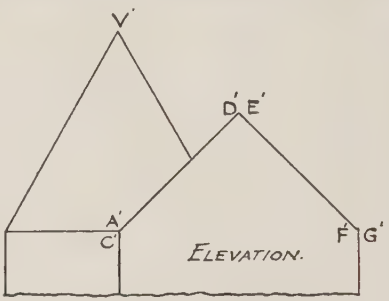
76. A conical roof, having a diameter of 10 ft. and an altitude of 18 ft., is cut back for a vertical pediment, the chord of the intersection being 7 ft. Draw the curve of intersection and give its name. (C.G.)

77. Fig. 19 gives outline plan and elevation of a plain pitched roof intersecting a conical roof.

Determine the true shape of the curb on the flat roof from which the timbers of the conical roof would spring. Scale $\frac{1}{4}$ inch = 1 foot.

(U.L.C.I.)

78. A conical roof 15 ft. high and 8 ft. diameter at the base is intersected by a straight vertical wall. The face of the wall is 2 feet from the centre of the roof. Determine the shape of the curb to be fixed to the wall to take the feet of the intercepted roof timbers, show how the normals for jointing the segments together



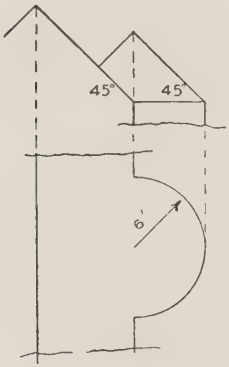


Fig. 19.

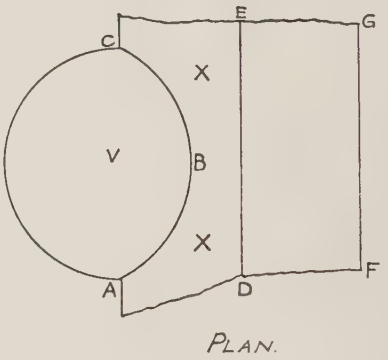


Fig. 20.

are determined, and give the geometrical name for the outline of the curb. Scale $\frac{1}{4}$ inch = 1 foot.

79. Fig. 20 illustrates two views of a conical roof intersecting an ordinary pitched roof. The plan of the intersection is not correctly shown in the diagram. Draw the elevation, and a plan showing the correct intersection. Develop the surface of the roof marked *X* in the lower diagram, and show the true shape of the intersection.

(D.A.I.)

ON CHAPTER XV

80. Set out the head of a solid door frame semicircular in elevation and segmental on plan. Span between rebates 3 ft. 6 ins., radius of plan curve 4 ft. Jambs out of 5 ins. × 3 ins. Scale 1 inch = 1 foot. The jambs are to be parallel and the head of the frame in two segments.

81. Determine the face moulds and bevels for a circle on circle frame with radiating jambs. Take any suitable scale and sizes.

82. Masonry arch in cylindrical wall. Project the views as shown and dimensioned, scale one-eighth. Also set out face moulds or patterns for the stone marked *A*. Fig. 21.

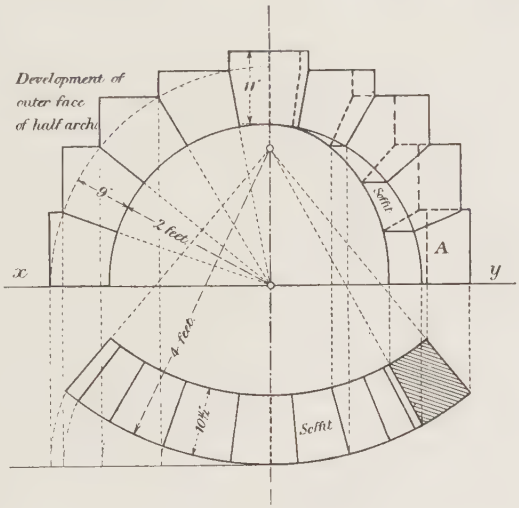


Fig. 21.

Explanation.—It will be seen that the shape of the outer cylindrical face of the arch is defined by the given development; moreover the drawing shows that the soffit is a warped surface generated by a horizontal line which when produced always intersects the vertical axis of the cylindrical surfaces, and that the horizontal edges of the stones all radiate from this axis. (B. o. E.)

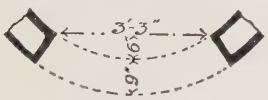


Fig. 22.

83. Fig. 22 is the plan of an opening to be covered by a segmental arch which has a 9-in. face and a rise of 12 ins. Show clearly how you would obtain the moulds in order to cut the voussoir next the springer. (C.G.)

84. Make drawings of the upper part of a semi-circular headed cased frame, 4 ft. 6 ins. wide between the styles; the plan to be circular and struck with a radius of 10 ft. Show the development of the face moulds. (C.G.)

85. The window described in Problem 84 has splayed linings 10 ins.

wide. Show the development of the linings, and explain the method of bending them. (C.G.)

86. A building between two streets formed at an acute angle to each other is to have a shop front formed with a circular end with a 6 ft. radius to outside of framing. At the angle the transom, 3 ins. deep, is to be given a segmental form in elevation, rising 1 ft. 3 ins. in a width of 6 ft. Develop one-half the face mould. Scale one quarter full size. (C.G.)

87. A circular porch has internal radius of 4 ft., walls $1\frac{1}{2}$ bricks thick, and an opening for doors 4 ft. wide, with radiating jambs and $\frac{1}{2}$-brick reveals. The doors are 7 ft. high, and above the transom the head is semicircular and filled with a fanlight and follows the plan of the wall. Draw to 1 inch scale half plan, half elevation, and vertical section, and make the necessary drawings for setting out the circular work. (C.G.)

ON CHAPTER XVI

88. The illustration, Fig. 23, shows the well-hole of a geometrical staircase with a veneered string. The rise is $6\frac{1}{2}$ ins. Make an elevation

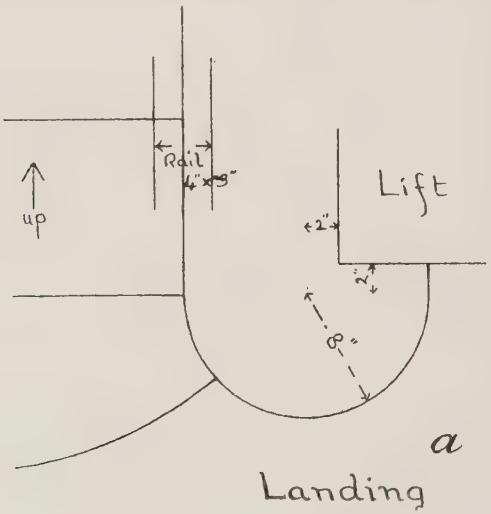


Fig. 23.

to $\frac{1}{2}$ in. scale of the development of the string. Make, $\frac{1}{4}$ full size, a plan of part of the string, showing the method of building up and of jointing the curved with the straight string. (C.G.)

89. Draw a handrail scroll for a rail 4 ins. × 3 ins. ; the scroll to be 12 ins. over all in width. Scale half full size. (C.G.)

90. Show all the necessary lines and moulds and bevels for the shank of the scroll referred to in the previous question, the scroll being over the bottom step of three commode steps in a quarter space. Draw the plan of the stairs, and state what rise and going you have adopted. Scale quarter full size. (C.G.)

91. It is necessary to insert two winders in a " half space " between two flights of stairs. The well hole is 2 ft. in diameter. Draw the plan of the stairs ; state the proportion you have adopted between the tread and riser ; and make complete drawings of the handrail moulds and bevels. Scale 2 ins. = 1 foot. (C.G.)

92. Develop the string and show the spacing of balusters in the previous question. (C.G.)

93. Fig. 24 gives the plan and part of the elevation of a portion of a handrail of rectangular cross-section (as prepared for moulding) ; complete the elevation of the rail. Set out the " face mould " for this rail. Draw also the " falling mould " for the central vertical section *CDDC* of the rail. (B. o. E.)

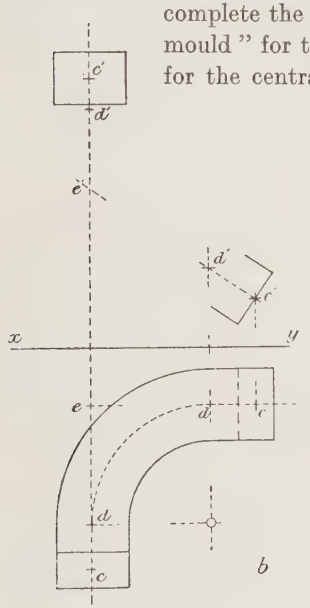


Fig. 24.

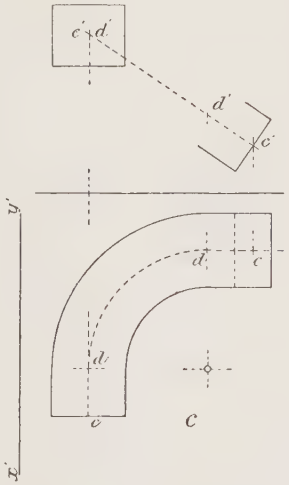


Fig. 25.

94. Fig. 25 gives the plan and part of the elevation of a portion of a handrail of rectangular cross-section (as prepared for moulding) ; complete the elevation of the rail. Draw a second elevation on x' y'. Set out the " face mould " for this rail. (B. o. E.)

95. Give all the necessary moulds and bevels for the handrail, as sketch (Fig. 26). Rise of stairs 6 ins., going 11 ins. (C.G.)

96. Show the geometrical setting out of the handrail wreath at the foot of the staircase shown in Fig. 27. The rise of the steps to be $6\frac{1}{2}$ ins., the going of the straight flight 11 ins. The handrail to be out of 4 ins. by 3 ins. ; the centres for the spiral are indicated. (C.G.)

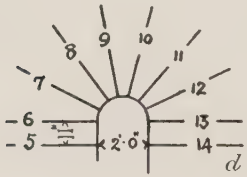


Fig. 26.

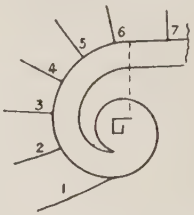


Fig. 27.

97. A geometrical staircase 3 ft. wide with a well-hole 1 ft. 9 ins. wide has a semi-circular end and around it eight steps, the treads in the other portion of the flight being 9 ins. wide and the risers $7\frac{1}{3}$ ins. high. Set up to a $\frac{3}{4}$-in. scale the development of the open string around the semicircular end.

ON CHAPTER XVIII

98. The moulding shown in Fig. 28 is that on the purlins of a conical roof ; this intersects with the rafter which is at right angles to them. What will be the section, supposing the purlins to be fixed

Fig. 28.

vertically, the angle of the roof to be 65°, and the diameter 10 ft. ? Illustrate the moulds required, and show how they should be applied in both types of purlins. Scale 4 ins. to the foot. (C.G.)

N.B.—The sketch given is not to scale.

99. The plan is given of two branches H, K of a cornice (Fig. 29). The branch H runs horizontally, K has an upward inclination of 30°, and the two have a common vertical section TT of the shape S.

Determine the cross-sections of H and K and the section of the latter by the vertical plane VV.

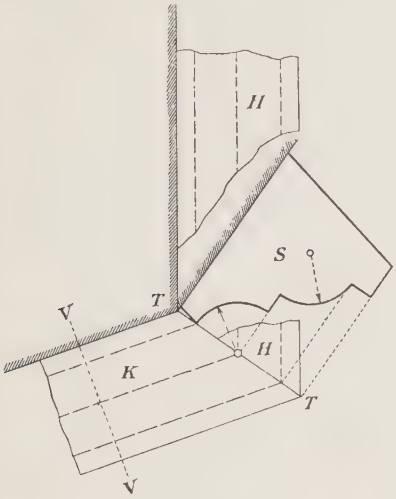


Fig. 29.

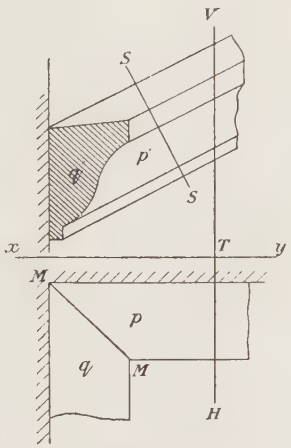


Fig. 30.

The mouldings being made of sheet metal, set out patterns for marking off the ends TT of the flat plates before they are bent.

(B. o. E.)

100. A moulding running horizontally is shown in plan and cross-sectional elevation at q, q' (Fig. 30). An inclined or raking moulding

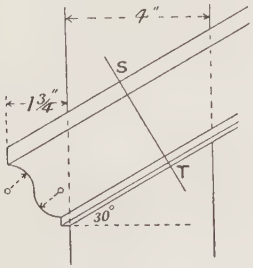


Fig. 31.

is given at p, p'. Set out the proper form of cross-section at SS for the latter in order that the two mouldings shall mitre correctly together

at the vertical joint MM. Find also the shape of the section of P by the vertical plane VTH. Set out the bevels or templates which would enable a workman to cut the mitre on P at the proper angle. (B. o. E.)

101. The elevation of a raking moulding returned on a pilaster is given in Fig. 31. Complete the elevation and determine the section on ST. (C.G.)

ON CHAPTER XIX

102. The circular arc ABC (Fig. 32) rotates about its chord AC. Draw the elevation and the plan of the figure generated.

Determine the shadow cast on the horizontal plane by parallel rays of light, one of which, R, is given. Show the margin of light and shade on the surface of the solid. (B. o. E.)

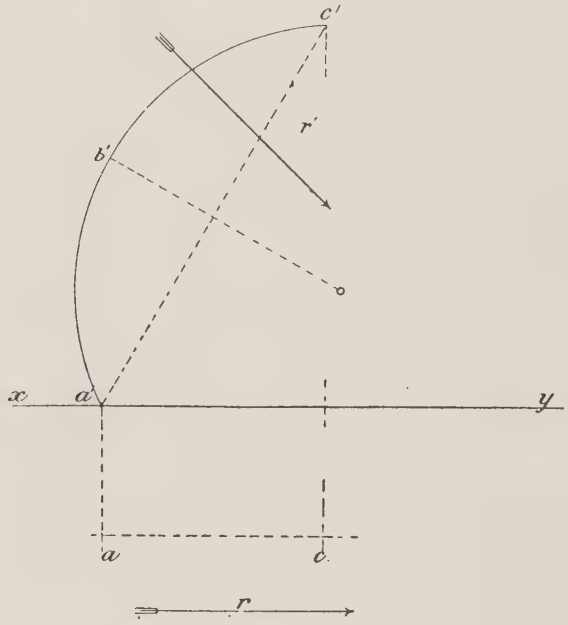


Fig. 32.

103. Bell buoy (Fig. 33). Determine the shadow cast by the buoy on the horizontal plane, the direction of the parallel rays of light being given.

Indicate, both in plan and elevation, what portion of the surface of the buoy is in shade. (B. o. E.)

104. Fig. 34 shows a hemisphere mounted on a square board; CC is a vertical rod. Project all the shadows that are cast on the spherical surface, board and planes of projection, the rays of light being parallel to the given ray R. Show also the portion of the curved surface which is in shade. (B. o. E.)

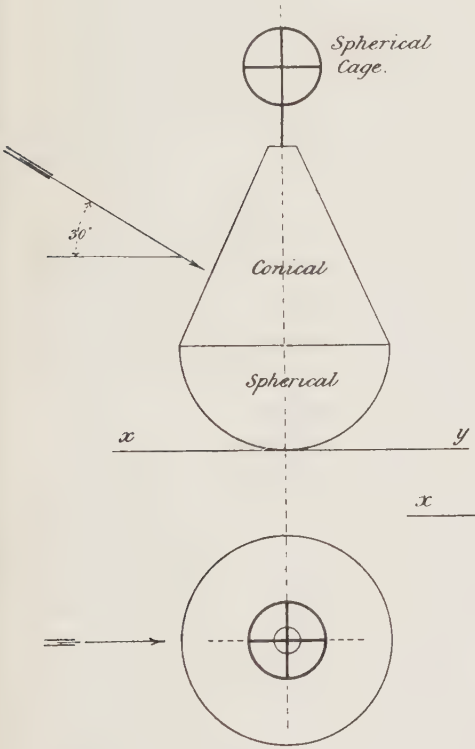


Fig. 33.

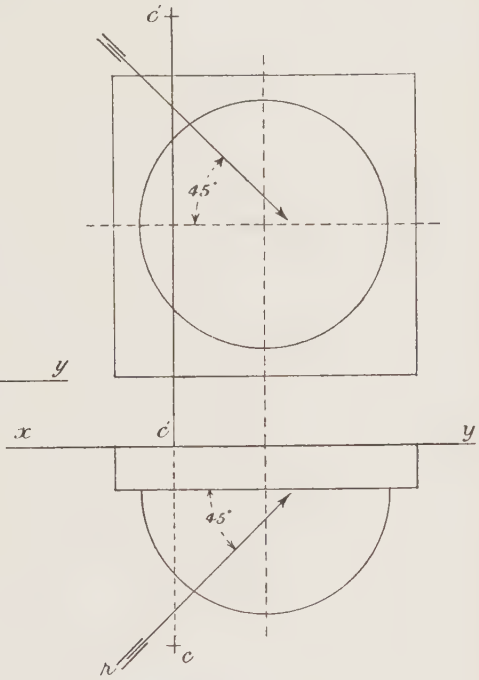


Fig. 34.

105. The plan and elevation of a semicircular niche are given (Fig. 35). Its domical ceiling is half a hemisphere with a semicircular eye or opening at the crown as shown.

If r, r' are the projected directions of parallel rays, show the elevation of the shadows cast inside upon the wall and ceiling, and also the plan of the shadows projected on the ground outside the niche. (B. o. E.)

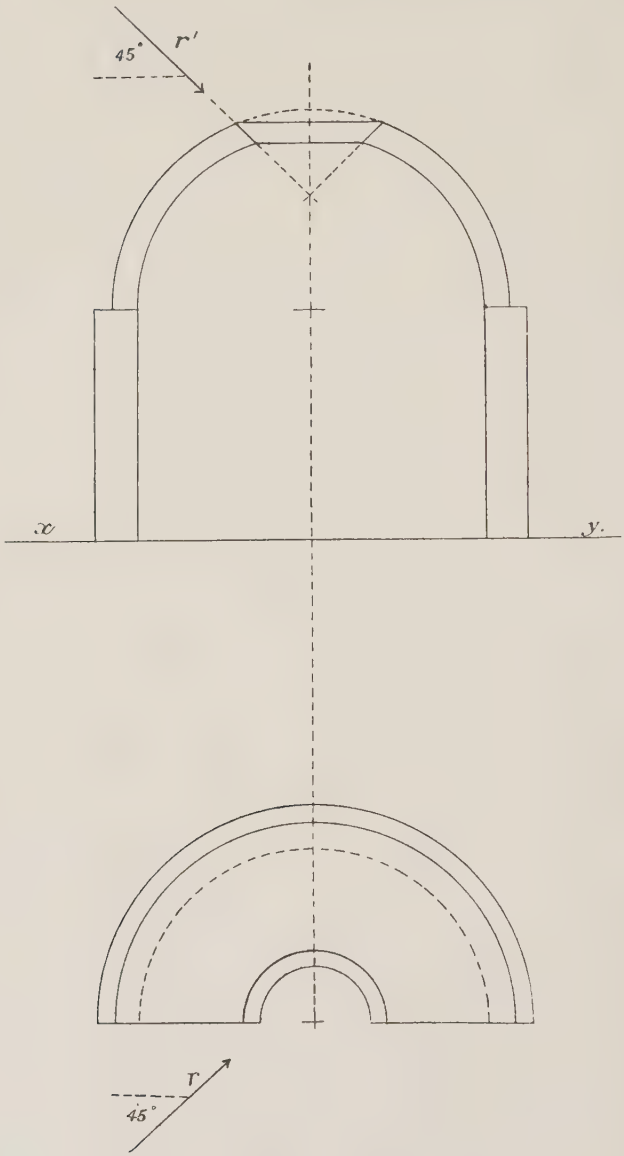


Fig. 35.

ON CHAPTER XX

106. Two elevations of a hut are shown (Fig. 36), and you are required to make a perspective view as follows :

The picture plane is to be in front of the hut, to contain the corner P, and to make 40° with PQ. The point of sight is to be opposite to P, at the same height, and at a distance from the picture-plane equal to $1\frac{3}{4}$ PQ.

N.B.—The two elevations need not be copied, dimensions being taken directly from the diagram. (B. o. E.)

107. Find in perspective a point A on the ground-plane 1 foot to the right of the spectator and 4 feet within the picture. From A draw

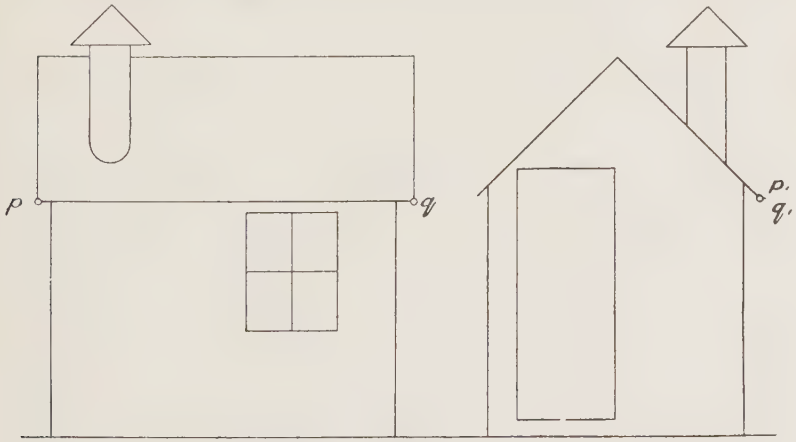


Fig. 36.

a line 7 ft. long, receding towards the right and inclined to the picture-plane at an angle of 30°. This line is to be the side of a square, lying on the ground-plane, A being its nearest corner. On the square as base erect a cube, and on the upper surface of the cube place a right pyramid having a base 11 ft. square, the edges of which are to be parallel to the respective edges of the upper surface of the cube. The pyramid is to be 6 ft. high, with its axis in the same straight line as the vertical centre-line of the cube. The eye is to be 4 feet above the ground and 11 feet from the picture. Scale $\frac{1}{2}$ inch = 1 foot.

[The horizon must be drawn across the long way of the paper, $5\frac{1}{2}$ inches from the top, and the centre of vision placed $6\frac{1}{2}$ inches from the left-hand edge of the paper.] (B. o. E.)

108. The plan and elevation of a brick porch are given (Fig. 37). Draw the porch perspectively under the conditions shown. (B. o. E.)

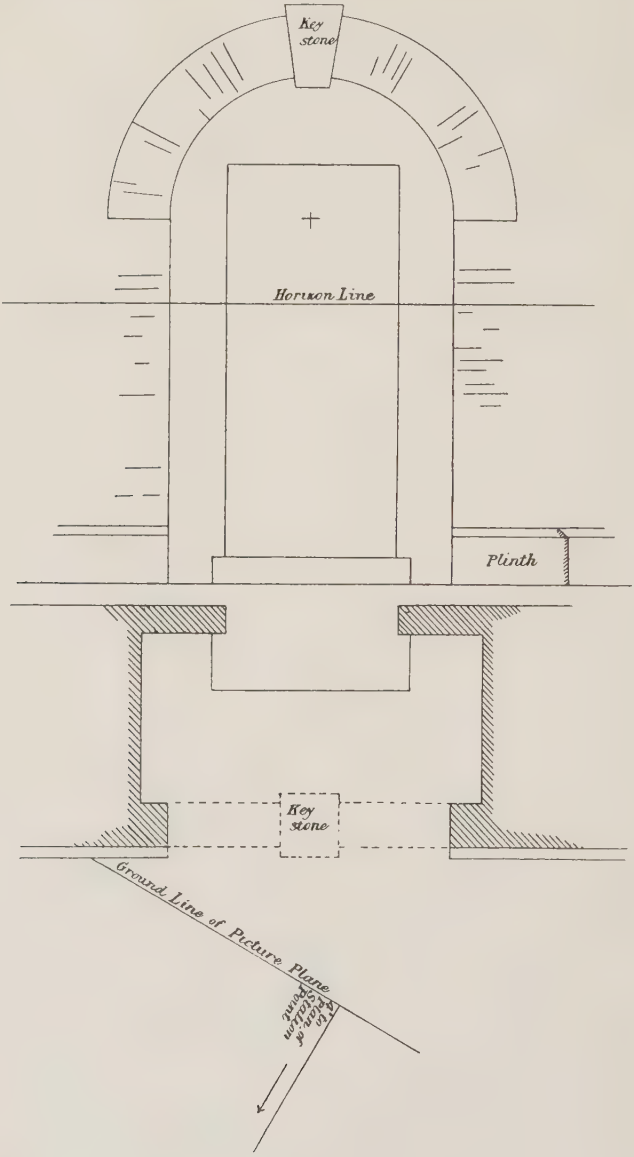


Fig. 37.

ON ROOFS

109. Two projections of a "hip" roof are given (Fig. 38). Find the angle of backing of the hip rafter *RR*—that is, the dihedral angle between the roof surfaces *A*, *B*. Determine the two bevels to which the upper end of a jack rafter *K* must be cut, so that it shall bed against the vertical face of the hip rafter.

Set out the corresponding bevels for the end of a purlin *P*.

(B. o. E.)

110. A roof having a pitch of 45° is to be formed with a hipped end over a building, the end of which is not at right angles to the sides (see sketch showing plan, Fig. 39). Show how you would determine the length of each of the hip rafters, the angle at the back and the bevels for purlins. (C.G.)

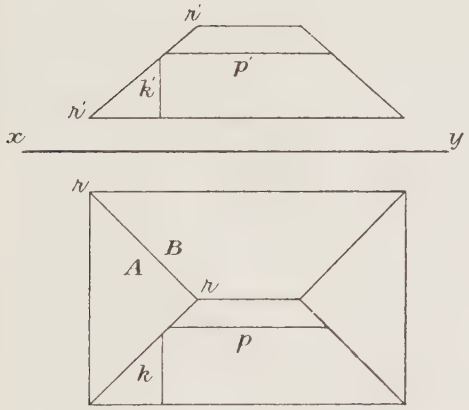


Fig. 38.

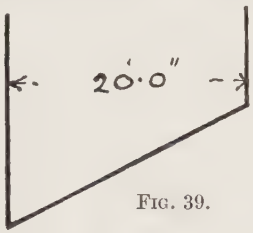


Fig. 39.

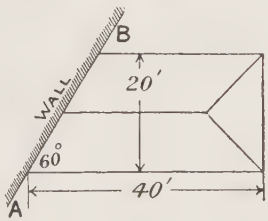


Fig. 40.

111. A roof finishing against a wall *AB* is shown in Fig. 40. Find lengths and "backing" of the "plates" fixed to wall to receive the ends of the rafters.

Show by sketches, with explanation, how the bevels and lengths of jack rafters, for both ends of one face of the roof, can be obtained. (C.G.)

112. The purlins and the hip are so arranged in a roof that a certain amount of purlin projects below the bottom edge of the hip. Show how you would obtain the necessary lip cut of the purlins under the hip. (C.G.)

ON DEVELOPMENT OF SURFACES

113. Fig. 41 shows the plan and elevation of the top of an oval-shaped tank. The top is to be made up in four pieces. Draw the required pattern for one piece. Scale 1 inch = 1 foot. (C.G.)

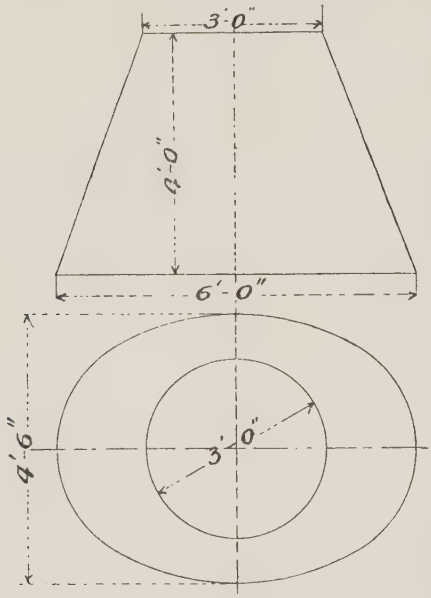


Fig. 41.

114. Two circular pipes connected at right angles are strengthened by gussets *G*, *G*, the surfaces of which are cylindrical with the generators parallel to *AB*.

Draw the given elevation, as dimensioned in Fig. 42, to a scale of $\frac{1}{8}$th full size. Add an end elevation and a plan, carefully projecting the curves.

Develop the surface of one of the gusset plates. (B. o. E.)

115. A cylindrical pipe passes through a funnel, as is shown in Fig. 43. Draw the pattern for one of the holes in the funnel. Scale 1 inch = 1 foot. (C.G.)

116. Fig. 44 represents a sheet metal pipe. Draw the plan of the section of the pipe made by the horizontal plane *SS*.

Obtain a development of the truncated oblique cone, showing the shape of the piece required to form the middle portion of the pipe. Omit all allowances for seams. (B. o. E.)

117. The diagram Fig. 45 shows a cylinder rising from a hemispherical dome. Determine the curve of intersection of the surfaces. Develop one of the symmetrical halves of the cylindrical surface. (B. o. E.)

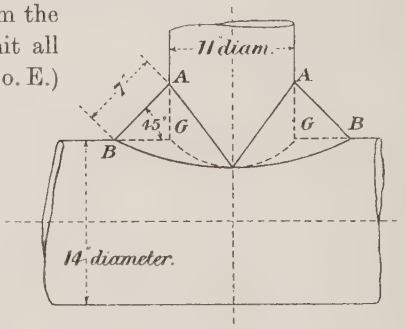


Fig. 42.

118. Elbow pipe for gutter, made in sheet metal (Fig. 46). Find the true shape of the elbow joint *AA*. Set out a template for use in making the gutter, that is, develop one branch *B* of the elbow. (B. o. E.)

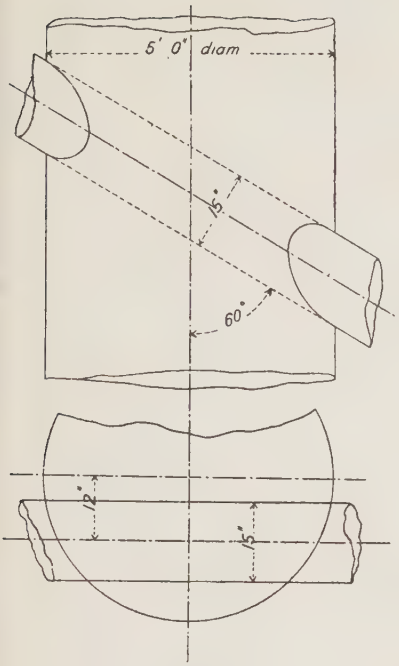


Fig. 43.

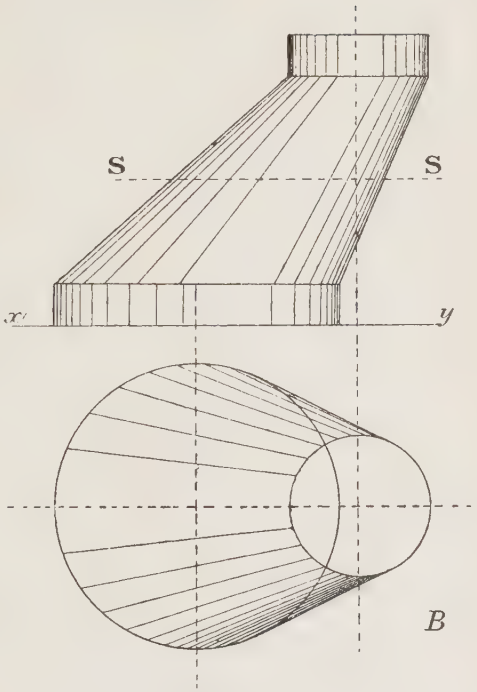


Fig. 44.

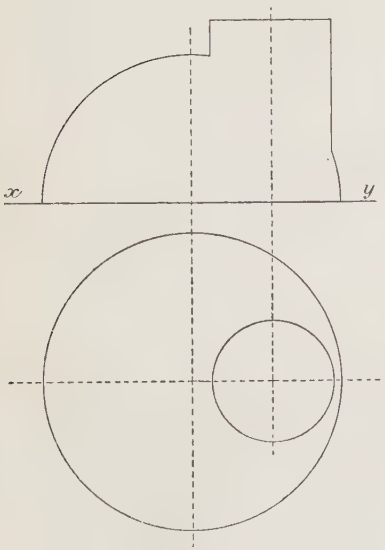


Fig. 45.

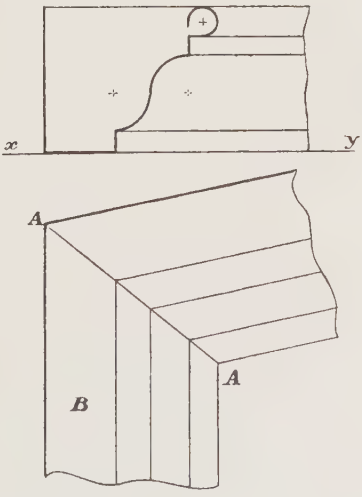


Fig. 46.

OTHER PROBLEMS

119. Fig. 47 shows the plan and two elevations of a bent pipe at the square angle of a wall.

Determine the real angle between the two branches of the pipe, *AB* and *BC*. (D.A.I.)

120. Brick sewer (Fig. 48). Set out the figure to a scale of 1 inch = 1 foot. Show clearly the construction for determining the centre *C*, and measure the radius *r*.

Mark the junctions of all the circular arcs. (B. o. E.)

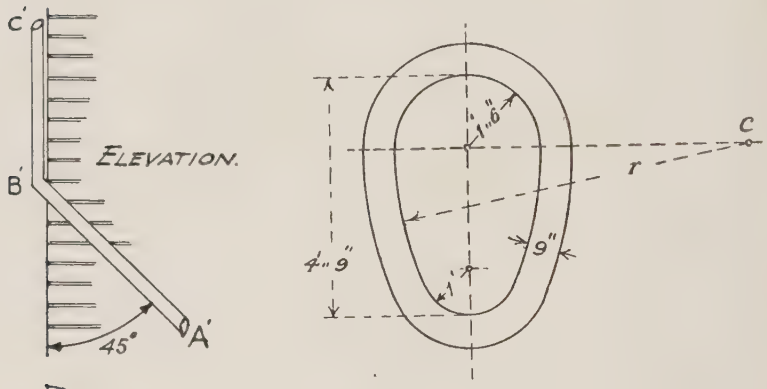


Fig. 48.

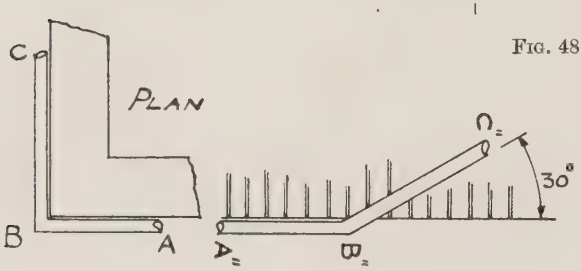


Fig. 47.

121. The lengths of the straight lines which join three points *A*, *B*, *C*, are known ; at any point *P* in the same plane as *A*, *B*, *C*, the angles *APC* and *BPC* are observed ; show how to find the distance of *P* from each of the points *A*, *B*, *C*. (B. o. E.)

122. Show how to inscribe, in a given triangle *ABC*, a triangle similar to another given triangle *DEF*. (B. o. E.)

123. The roof of a summer-house is octagonal in plan, the section of the roof at right angles to the wall plate is shown in Fig. 49 : make a drawing showing the shape of the hip rafters and how the backing is obtained. (C.G.)

124. The moulding given in Fig. 50 represents a horizontal section through a stone jamb, finishing on the soffit of a semicircular

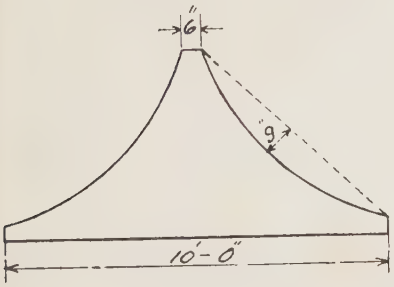


Fig. 49.

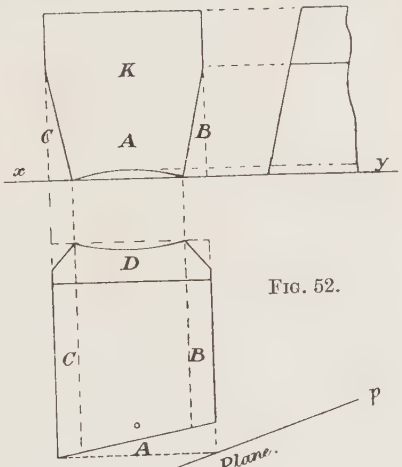


Fig. 52.

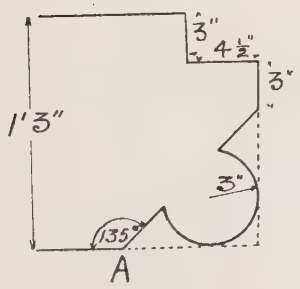


Fig. 50.

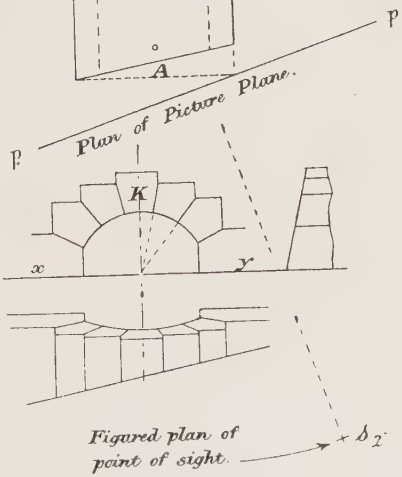


Fig. 51.

arc springing from the elevation of the arris *A*, the radius of the head being 3 ft. and the clear span 4 ft. 6 ins.

Draw a vertical section of a plane at right angles to the face of the arch, showing the intersection between the moulding and the soffit. Scale 1 inch = 1 foot. (U.L.C.I.

125. Masonry arch (Fig. 51). The keystone K, enlarged, is shown in projection. Set out the true shapes of the faces A, B, C, and D. Find the dihedral angle between the faces C and D. (B. o. E.)

126. The dotted lines (Fig. 52) show a rectangular block of stone, out of which the voussoir K is to be cut. Represent this block, with the voussoir enclosed, in perspective projection. The picture-plane PP and the point of sight S are specified in the drawing.

Show hidden edges by dotted lines. (B. o. E.)